AMS Spring Western Sectional Meeting
April 4, 2004 (USC campus)

"Two-sided barrier problems with jump-diffusions"
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No preprint yet, but overheads to be posted at www.optioncity.net (Publications)
(This work is preliminary and incomplete.)

Objectives:

1. Develop analytic formulas for Green function's for certain two-sided barrier/exit problems,

where the underlying stochastic process:
$\{X_t\}$ = 1D jump-diffusion (Levy process)

2. Develop simple alternative numerical methods to check the results.

Connection with Finance

Suppose a stock price S_t follows an exponential Levy process:
$S_t = S_0 \exp(X_t)$,
(Under a martingale pricing measure Q)

<u>Properties of Levy class</u>:
(i) independent increments:
$X_t - X_s$ **and** X_s **are independent** $(t > s)$
(ii) time-homogeneous (stationary):
Distribution of $\{X_{t+s} - X_s\}$ **does not depend on s.**
(iii) continuous (in probability) at any fixed time:
The jumps occur at unpredictable times.

<u>Levy sub-class: the jump diffusions</u>:

$$X_t = \omega t + \sigma B_t + \Delta X_t,$$

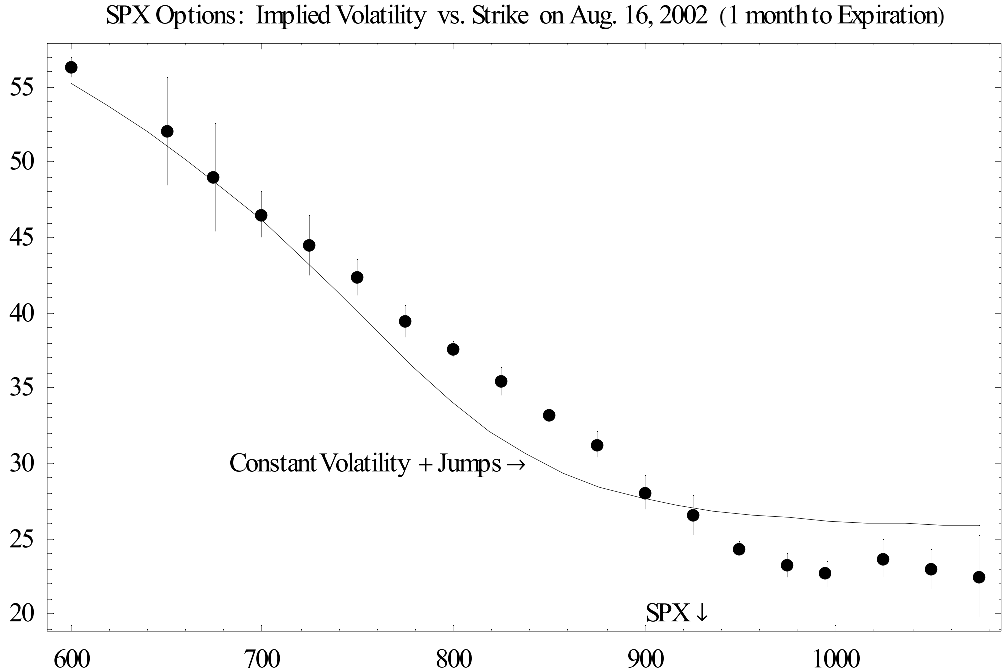
$B_t =$ **std. Brownian motion;** $\quad \Delta X_t = \sum_{i=1}^{N_t} \xi_i$

(i) jump events N_t follows a Poisson process:
Jump probability(t) $\approx \lambda \Delta$t , $\lambda =$intensity

(ii) jump amplitudes ξ are then drawn independently:
$\xi \sim F(\xi)$ **(Jump size density)**

Connection with Finance (cont.)

Figure 1



Example Jump-diffusion model:
Merton's 1976 jump-diffusion model:

Jump sizes: $$F(\xi) = \frac{1}{\sqrt{2\pi\sigma_J^{\ 2}}} \exp\left[-(\xi - \mu_J)^2 / 2\sigma_J^{\ 2}\right]$$

Typical S&P500 Index Option Smile Fit

$$\lambda \approx 0.3, \quad \mu_J \approx -0.25, \quad \sigma_J \approx 0.10$$

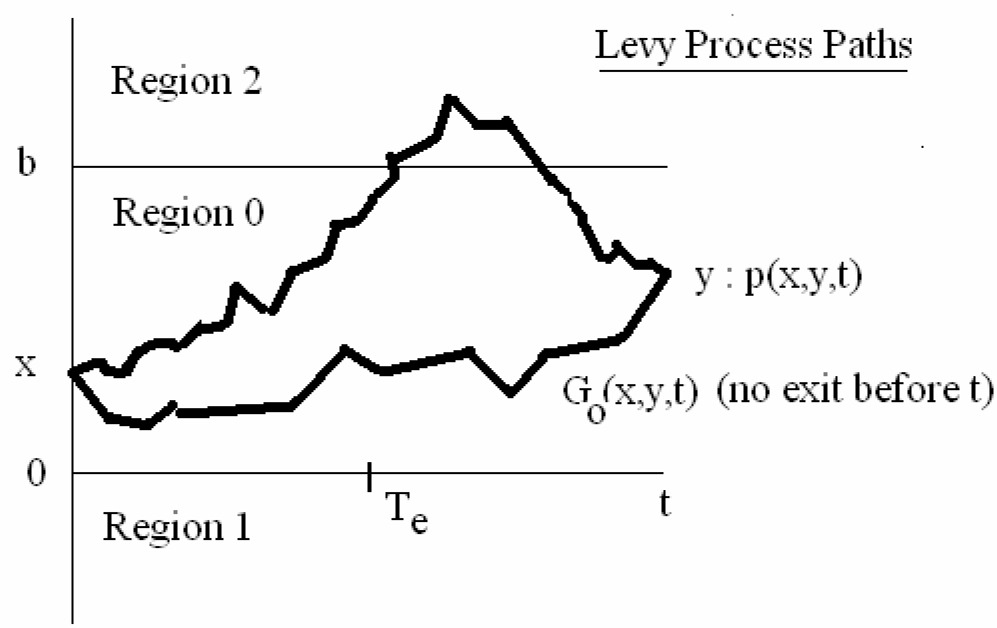
<u>Plain vanilla options</u> like these are easily priced under <u>any</u> Levy/jump-diffusion process. (single integral transform)

<u>Exotics</u>: single barrier options; more difficult, but formulas are known. (Integrals using Wiener-Hopf factors $\phi_q^{\pm}(z)$)

Double barrier: <u>No general formulas for G are known</u>

$$OptionValue(x) = e^{-rT} \int_0^b G(x, y, T) Payoff(y) dy$$

The Two-sided Exit Problem



$$X_0 = x; \quad T_e = \textbf{ first exit from } I = (0,b).$$

Some probability densities: : | **Support for y**

Unrestricted law:

$P_x(X_t \in dy) = p(x,y,t)dy.$ $\quad (-\infty,\infty)$

(Conditional) Green function:

$P_x(X_t \in dy, T_e > t) = G_0(x,y,t)dy.$ $\quad$ **I = (0,b)**

Exit densities:

$$P_x(X(T_e) \in dy, T_e \in dt) \qquad I_c = (-\infty,0) \cup (b,\infty)$$

$$= G_e(x,y,t)dydt$$

$$= [G_1(x,y,t) + G_2(x,y,t)]dydt$$

The Markov Property + Conservation of Probability

$$p(x,y,t) = G_0(x,y,t) + \int_{I_c} d\xi \int_0^t ds\, G_e(x,\xi,s)\, p(\xi,y,t-s)$$

But, for a Levy process: $p(x,y,t) = \tilde{p}(y-x,t)$

$$\tilde{p}(y-x,t) = G_0(x,y,t) + \int_{I_c} d\xi \int_0^t ds\, G_e(x,\xi,s)\, \tilde{p}(y-\xi,t-s)$$

Next:

1. **Take Fourier transform of both sides** $\int_{-\infty}^{\infty} dy\, e^{izy}(...)$
2. **Take Laplace transform of both sides** $\int_0^{\infty} dt\, e^{-qt}(...)$
3. **Use** $\int_{-\infty}^{\infty} e^{izy} p(y-x,t)dy = e^{izx - t\psi(z)}$,
   where $\psi(z) =$ **characteristic exponent of Levy process**

Define $Q_i(z) = \int dt\, e^{-qt} \int dy\, e^{izy} G_i(x,y,t), \quad i = 0,1,2.$

$\Rightarrow$

Fundamental Transform Identity: (FTI)

$$\frac{e^{izx}}{q+\psi(z)} = Q_0(z) + \frac{Q_1(z)+Q_2(z)}{q+\psi(z)}$$

Kemperman, "A Wiener-Hopf Type Method for A General Random Walk with a Two-sided boundary ." (Ann. Math. Statistics 1963)

His starting point is the FTI:

$$\frac{e^{izx}}{q+\psi(z)} = Q_0(z) + \frac{Q_1(z)+Q_2(z)}{q+\psi(z)}\ , \qquad x \in (0,b)$$

(The dependence upon x in $Q's$ is implicit/suppressed)

His focus is the discrete-time Random Walk:

$$X_n = \xi_1 + \xi_2 + \cdots \xi_n \ \ ; \quad \xi \sim F(\xi).$$

He shows how to find Q_0 when

$$F(\xi) = c\,e^{\eta\xi} 1_{\{\xi<0\}} + G(\xi) 1_{\{\xi\geq 0\}}, \quad G \ \underline{\text{arbitrary}}.$$

My goal is to generalize this to the Continuous-time case:

$$X_t = \omega t + \sigma B_t + \sum_{i=1}^{N_t} \xi_i\,, \quad \xi_i \sim F(\xi)$$

Ingredients to the argument

1. Introduce the WH Factors ($z \in$ horiz. strip about origin)

$$\frac{q}{q+\psi(z)} = \phi_q^+(z)\phi_q^-(z)$$

Properties: $\phi_q^+(z)$ is a characteristic function of a probability distribution with support on $(0,\infty)$.

2. Introduce $M(a,b)$, a class of functions with support in (a,b). Their Fourier transforms are, by definition, in the class $\hat{M}(a,b)$. For example, $\phi_q^+(z) \in \hat{M}(0,\infty)$.

3. Introduce the truncation-by-A operation: ($\hat{f}$ =Fourier transform of f)

$$[\hat{f}]^A = [\hat{f}]^A(z) = \int_A e^{izx} f(x)dx$$

Steps in the argument

FTI: $$\frac{qe_x}{q+\psi(z)} = qQ_0 + \frac{q}{q+\psi(z)}(Q_1+Q_2); \qquad e_x = e^{izx}$$

WH factors: $$qQ_0 = e_x\phi_q^+\phi_q^- - (Q_1+Q_2)\phi_q^+\phi_q^-$$

$$q\frac{Q_0}{\phi_q^-} = e_x\phi_q^+ - (Q_1+Q_2)\phi_q^+ \qquad (1)$$

Then, Kemperman proves, and I rely upon:

A. $Q_2\phi_q^+ \in \hat{M}(b,\infty)$

B. $\dfrac{Q_0}{\phi_q^-} \in \hat{M}(-\infty,b)$

Apply the truncation operator $[\cdots]^{(-\infty,b)}$ to both sides of (1). This knocks out $Q_2\phi_q^+$ since its transform = 0 on $(-\infty,b)$.

$$\Rightarrow q\frac{Q_0}{\phi_q^-} = [e_x\phi_q^+]^{(-\infty,b)} - [Q_1\phi_q^+]^{(-\infty,b)}$$

Steps in the argument (cont.)

$$q\frac{Q_0}{\phi_q^-} = [e_x\phi_q^+]^{(-\infty,b)} - [Q_1\phi_q^+]^{(-\infty,b)}$$

Example truncation calculations for the right-hand-side

Let $U(z) = e_x(z)\phi_q^+(z) = \int_{-\infty}^{\infty} e^{izy}u(y)dy.$

Recall $[U]^A(z) = \int_A e^{izy}u(y)dy \Rightarrow$

$$[U]^{(-\infty,b)}(z) = \int_{-\infty}^{b} dy e^{izy} \int \frac{d\xi}{2\pi} U(\xi)e^{-i\xi y}$$

$$= \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} U(\xi)e^{ib(z-\xi)}$$

or

$$[e_x\phi_q^+]^{(-\infty,b)}(z) = e^{ibz} \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} \frac{e^{i\xi(x-b)}}{z-\xi} \phi_q^+(\xi)$$

Steps in the argument (cont.)

Recall that $Q_1(z)$ is the Fourier transform of the exit density for the lower barrier at 0. There are two ways to exit at the lower barrier:

1. A continuous touch at $X = 0$ due to the Brownian motion component, or

2. A negative jump from some point above the barrier, with jump amplitude drawn from $c\,e^{\eta\xi}1_{\{\xi<0\}}$.

Experience calculating with purely exponential jump models, leads to 'overshoot' (exit) densities which are <u>also</u> exponential with the same upside/downside parameters as the jump amplitude distribution.

Even though our upside jump distribution is <u>not exponential,</u> experience suggests the downside "solution ansatz:"

$$G_1(x,y,t) = A(x,t)\delta(y) + B(x,t)e^{\eta y}; \quad (y \le 0)$$

$$\Rightarrow Q_1(z) = \hat{A}(x,q) + \frac{\hat{B}(x,q)}{z - i\eta}, \qquad \mathrm{Im}\, z < \eta.$$

This has to be proved a self-consistent choice! (Gap #1)

Final steps in the argument

At this point, we have
(suppressing some dependencies on x and q)

$$qQ_0(z) = \phi_q^-(z)e^{ibz}H(z), \text{ where}$$

$$H(z) = \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} \frac{\phi_q^+(\xi)}{z-\xi} \left\{ e^{i\xi(x-b)} - A\, e^{-i\xi b} - \frac{B}{\xi - i\eta} e^{-i\xi b} \right\}$$

It can be shown (Mordecki, 2002),
for a mixed exponential + arbitrary jump density, that:

$$\phi_q^-(z) = \left(\frac{z-i\eta}{-i\eta}\right)\left(\frac{-i\alpha_{1q}}{z-i\alpha_{1q}}\right)\left(\frac{-i\alpha_{2q}}{z-i\alpha_{2q}}\right),$$

Where $(\alpha_1, \alpha_2) =$positive reals
$(i\alpha_1, i\alpha_2) =$ purely imag. roots of $q + \psi(z)$ in upper z-plane.

But $Q_0(z)$ is the transform of a prob. density with support in $(0,b)$. This transform is analytic in $\operatorname{Im} z > 0$. So, the two poles in $\phi_q^-(z)$ must be cancelled zeros in the integral $H(z)$ at $z = (i\alpha_1, i\alpha_2)$. That is

$$H(i\alpha_1) = 0 \text{ and } H(i\alpha_2) = 0$$

$\Rightarrow$ Two conditions that determine the unknowns (A,B)

Final Formulas (The Answer)

Define:

$$I_1(x,z) = \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} \frac{e^{i\xi(x-b)}}{z-\xi} \phi_q^+(\xi)$$

$$I_2(z) = \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} \frac{e^{-i\xi b}}{(z-\xi)(\xi - i\eta)} \phi_q^+(\xi)$$

Determine A, B from the two linear relations:

$$A(x) I_1(0, i\alpha_j) + B(x) I_2(i\alpha_j) = I_1(x, i\alpha_j), \quad \text{j} = 1,2$$

Then,

$$\begin{aligned}\hat{G}_0(x,y,q) &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} Q_0(z) e^{-izy} \\ &= \frac{1}{q} \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{iz(b-y)} \phi_q^-(z) H(z),\end{aligned}$$

where

$$H(z) = \int_{\operatorname{Im}\xi > \operatorname{Im} z} \frac{d\xi}{2\pi i} \frac{\phi_q^+(\xi)}{z-\xi} \left\{ e^{i\xi(x-b)} - A\, e^{-i\xi b} - \frac{B}{\xi - i\eta} e^{-i\xi b} \right\}$$

There is an ambiguity in the defn. of the I_2 contour. Is $\operatorname{Im}\xi < \eta$ or $> \eta$? I force it to always lie below, because this achieves numerical agreement with my lattice method. (This needs to be clarified: Gap #2).

Computations (Alternative Numerical Method).

I compare results for

$$\hat{G}_0(x,y,q) = \int_0^{\infty} e^{-qt} G_0(x,y,t)dt$$

vs. a simple explicit lattice algorithm. (Lewis, 2004).

Briefly: Divide the interval (0,b) into N points of size Δx. Choose $\Delta t = (\Delta x / \sigma)^2 / 3$, which corresponds to a well-known trinomial lattice method in finance.

To compute $G_0(x,y,t)$, start with

$$v_{old}(x_i) = G_0(x_i, y, 0) = \delta_{x(i),y} \text{=Point mass at y.}$$

Iterate recursively: $v_{new} = P\, v_{old}$, where P is an $N \times N$ (banded/Toeplitz) transition matrix:

$$P = \begin{pmatrix} p_0 \; p_1 \; p_2 & \cdots & p_{N-1} \\ p_{-1} \; p_0 \; p_1 & \cdots & p_{N-2} \\ & \cdots & \\ p_{-N+1} & \cdots & p_{-1} \quad p_0 \end{pmatrix}$$

Where $p_i = (1-\lambda\Delta t)\, p_i^{bm} 1_{|i|\le 1}$

$$+\lambda \Delta t\, G(x_i - .5\Delta x, x_i + .5\Delta x) 1_{|i|\ge 1}$$

Now $G = \int_{-\infty}^{x} F(y)dy$ is the jump-size distribution, (not density), and $p_0^{bm} = 2/3, \qquad p_{\pm 1}^{bm} = 1/6 \pm (\omega/\sigma)\sqrt{\Delta t / 12}$

Computations (Results)

Interval $I = (0,2)$, $y = 1$, various x
Parameters: $q = 0.5$, $\sigma = \sqrt{2}$, $\lambda = 3$, $\eta = 3$.
Jump size density: (up point jump + down exponential):

$$F(x) = .5\,\delta(x - 0.25) + 0.5\,\eta e^{\eta x} 1_{\{x<0\}}$$

Results: $\underline{\hat{G}_0(x, y = 1, q = 0.5):}$

Explicit Lattice: ($T_{\max} = 5$, no changes if $T_{\max} = 10$).

$N+1$	$x = 0.25$	0.5	0.75	1.0	1.25	1.5	1.75
120	0.0757	0.1626	0.2662	0.3954	0.2990	0.2058	0.1084
160	0.0758	0.1627	0.2663	0.3956	0.2991	0.2058	0.1084

Analytic Formula: $\xi_{\max} = z_{\max}$,
Double Integration; Mathematica AccuracyGoal =AG, AG+3.

AG, $z_{\max}$	$x = 0.25$	0.5	0.75	1.0	1.25	1.5	1.75
3,400	0.0743	0.1638	0.2662	0.3950	0.2988	0.2067	0.1068
5,400	0.0760	0.1630	0.2667	0.3952	0.2994	0.2059	0.1084

Summary: The numerical results encourage me to believe the final formulas are correct. Further work is needed to fill in the gaps in their derivation.

References

J.H.B. Kemperman, ***A Wiener-Hopf Type Method for A General Random Walk with a Two-sided boundary*****. Annals of Math. Statistics 34, 1963, 1168-93.**

Alan L. Lewis, ***Double Barrier Options with Jumps – a Simple "Universal" Algorithm*****, Wilmott magazine, Mar., 2004, 34-36.**

Ernesto Mordecki, ***Optimal stopping and perpetual options for Levy processes*****, Finance & Stochastics, VI, 2002, No. 4, 473-493.**

Note added after the talk: the full analytic solution, with the gaps filled in, will be given in:

Alan L. Lewis, ***Option Valuation under Stochastic Volatility, Vol. II: Jumps and Exotics*****, Finance Press (Newport Beach), 2004 forthcoming.**