Applications of Eigenfunction Expansions
in Continuous-Time Finance

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Abstract

We provide exact solutions for two closely related valuation problems in continuous-time finance. The first problem is to value generalized European-style options on stocks that pay dividends at a constant dollar rate. The second problem is to find the yield curve associated with the economy of R.C. Merton's "An Asymptotic Theory of Growth Under Uncertainty". In Merton's economic growth model, the interest rate process has a volatility linear in the rate level and a linear/quadratic drift. Both problems are solved by an eigenfunction expansion technique. The main technical difficulty is handling the problem of payoff functions which are not square-integrable with respect to the natural weight function of the models.

Key Words: European-style options, yield curve, eigenfunction expansions, R.C. Merton, bankruptcy, exit boundary, constant dollar dividend.

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1. INTRODUCTION

This paper presents exact solutions for two diffusion models for certain security prices in a perfect market. Both models were introduced and partially solved by Merton over two decades ago, which makes them relatively old problems of continuous-time finance. The first problem is the valuation of a European-style option on a stock that pays a constant dollar dividend rate. Merton (1973) found the time independent solution as the time to expiration becomes large. The second problem is to find the term structure of interest rates associated with a risk-adjusted process that has a volatility proportional to the rate level and a mean-reverting drift given by the sum of a linear and quadratic term. Merton (1975) derived this interest rate process as a consequence of a certain economic growth model and solved for the steady state probability distribution for the rate. In the first model, our new result is the calculation of the full time dependent option solution. In the second model, our new result is the determination of the yield curve. So, in both cases, while Merton found a time independent solution, we are providing the full dynamics.

As it turns out, the two models are closely related. The models are solved here by the technique of a Laplace transform of the time dependence which leads to eigenfunction expansions involving confluent hypergeometric functions. There are two new items here. First, the general technique of an eigenfunction expansion has, to the author's knowledge, received little attention in finance. Because this technique is very powerful, it may assist in the solution of other difficult problems. So, we have a new application of the general technique. Second, the nature of the initial conditions in these problems leads to a difficulty not discussed in the approach of standard treatises on this subject, such as Titchmarsh (1962). Briefly, in the approach of Weyl (1950), Titchmarsh and others (especially applications to quantum mechanics), the general theory is usually closely associated with the Hilbert space of square-integrable functions. But, in the finance problems of the type discussed here, sometimes the (initial and time-developed) payoff functions are in such a space and sometimes not. When they are not, this introduces certain additional eigenvalues into the problem which determine the large-time behavior of the solutions. As a result, the correct treatment of this situation makes our expansions not simply a routine exercise in the application of a standard mathematical technique.

Our results for the two models provide some insights on the important role played by a principal eigenvalue in the large-time dynamics of models in continuous-time finance. For example, in the interest rate term structure model discussed here, we can see how an increase in volatility can prevent equilibrium. Interest rate equilibrium is characterized by a strictly positive principal eigenvalue, a bond price that tends asymptotically to zero, and a strictly positive asymptotic yield. In contrast, non-equilibrium is characterized by a zero-value principal eigenvalue, a bond price that tend asymptotically to a strictly positive constant, and a zero-value asymptotic yield.

This paper is organized as follows. In Section 2, we provide the solution of the problem of options on stocks with a constant dollar dividend rate. In Section 3, we develop the term structure of interest rates for the interest rate process discussed above. In each section, while we briefly discuss applications and present examples, we believe that the main contributions are the solution proofs presented in Appendixes I and II.

2. OPTIONS ON STOCKS WITH A CONSTANT DOLLAR DIVIDEND RATE

2.1 Problem Definition.

The problem discussed in this section is to value certain generalized European-style claims on stocks. The random stock price at time *T*, denoted S_T, is assumed to follow a stochastic differential equation with a constant price return variance rate σ^2, but with an arbitrary drift rate. An owner of the stock receives a continuous payout at the constant rate of *D* dollars per year. The claims expire at time T^* with a specified payoff $f_0(S_{T^*})$. Prior to their expiration, the claims have no payouts. We assume perfect frictionless markets and the existence of a money market account with a constant riskless interest rate r. Using a no-arbitrage argument, the value of the option or generalized claim price $f(S,\tau)$ can be shown to satisfy the second order differential equation (Model I):

$$\text{(2.1)} \qquad \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + (rS - D)\frac{\partial f}{\partial S} - rf = \frac{\partial f}{\partial \tau}$$

where $\tau = T^* - T \geq 0,\ \ 0 \leq S < \infty$ and with the specified payoff at $\tau = 0$. Merton (1973, eq. 46) found the time independent $\tau \to \infty$ solution for the special case of the call option payoff: $f_0(S_{T^*}) = \max(S_{T^*} - K, 0) = (S_{T^*} - K)^+$, where K is the strike price. Our new findings include (i) the full time dependent put and call option solutions and (ii) the solution for *general* payoff functions in a certain function space. In addition, we clarify to what extent it makes sense to say that the dividend rate is constant, given the non-zero probability of the stock price reaching zero (bankruptcy). Equation (2.1) represents one of the simplest generalizations of the original Black-Scholes option theory to a world with continuous payouts. The other natural continuous payout is a continuous dividend *yield* δ. With a continuous yield, the dividend rate *D* above is replaced by $\delta \cdot S$ and eq. (2.1) has well-known option solutions, obtained by a transformation to the heat equation.

In terms of applications, Model I solutions with an option payoff provide an approximation to the value of European-style options on broad-based equity indexes. Over typical holding periods of a year or less, a constant dollar dividend rate approximation is generally better than a constant yield approximation since companies are slow to change their dollar dividends on this time scale. In addition, the diversified nature of many indexes makes the continuity assumption a reasonable approximation.

2.2 Boundary Conditions.

In the differential equation (2.1), both $S = 0$ and $S = \infty$ are singular points. In general, the ability and/or the necessity to specify boundary conditions at singular points depends upon the nature of the boundary. One can easily remove the third term $-rf$ from eq. (2.1) by introducing a discounting factor. Then, the equation is transformed into one suitable for Feller's boundary classifications [see Karlin and Taylor (1981), Chapt. 15]. These classifications are determined entirely by the behavior of the underlying (risk-neutral) stochastic process near the boundaries. One is first tempted to write the stochastic differential equation for the stock price dynamics as $dX = (rX - D)d\tau + \sigma X dz$, starting at $X(0) = S$. However, as shown below, this needs to be refined at $X = 0$.

In this model, nothing novel for finance occurs at infinity, which is a *natural* boundary and has zero probability of being reached by the stock price dynamics in *finite* time. In general, this prevents the specification of boundary conditions at infinity. This is the same boundary behavior which occurs in the Black-Scholes theory for a non-dividend paying stock. Clearly, this boundary behavior is expected because, as the stock price increases, a constant dividend becomes negligible.

The behavior near the origin is more interesting; the origin is an *exit* boundary, attainable in finite time from any finite starting value $X(0) = S \in [0, \infty)$. A simple argument for finite time is as follows: once a random jump takes the stock price to a small value of X, then the process behaves like $dX \approx -D\, d\tau$, which will be attracted to the origin as long as $D > 0$. In Section 2.9, we calculate the exact probability that the origin is reached from the interior. If the stock price reaches zero, an economically reasonable assumption is (i) the stock price process stops, meaning that absorption occurs (the price remains zero subsequently) and (ii) the dividend payments stop also. While this is not the only possible boundary behavior, it is a natural choice for this problem and is compatible with the Merton large-time solution. The absorption event itself can be sensibly termed *bankruptcy.* The possibility of bankruptcy is an interesting feature in this model in contrast to the Black-Scholes model with a continuous yield. In the Black-Scholes case, the origin is a natural boundary, unattainable in finite time from a strictly positive initial stock price. Bankruptcy also can occur in Cox's Constant Elasticity of Variance Model (Cox and Ross (1976)). In the Cox model, the bankruptcy is caused by the behavior of the volatility term, while here it is caused by a cash flow in the drift term, which is somewhat more natural. Since the dividend payments stop if bankruptcy occurs, a more complete description of the underlying stochastic process is $dX = (rX - \mathcal{D}(X))d\tau + \sigma X dz, \quad X(0) = S$. Now $\mathcal{D}(X)$, the dividend process, is given by the time homogeneous function $\mathcal{D}(X) = D, \ (X > 0)$ and $\mathcal{D}(X) = 0, \ (X = 0)$.

As a practical matter, one would expect a real-world company to reduce or eliminate its dividend, if possible, if its stock price were very low. If the "stock" were actually a bond, this might not be possible contractually and a real-world bankruptcy similar in spirit to the idealized one here could result anyway. As a generalization of the model here, one might consider more complex dividend policies $\mathcal{D}(X)$ to better model real-world behavior. Whether or not these more complex policies could lead to the security price reaching zero may be investigated by the Feller boundary criteria. Mathematically, the question would be whether or not the origin is an exit boundary. Calculating option values would presumably require a numerical approach to the valuation equation (2.1).

2.3 Impact of Bankruptcy on Option Values.

Clearly, the possibility of bankruptcy may have a significant impact on the value of a put option $P(S,\tau)$, with new aspects not seen in the Black-Scholes theory, as we shall show in detail in section 2.11. To briefly see the impact, consider the put value as the discounted expectation $P(S,\tau) = \exp(-r\tau)E_S[(K - X_\tau)^+]$. (Notation: when we need to show a time argument for X, we write $X(\tau)$ or X_τ interchangeably.) Here $E_S[f(X_\tau)]$ denotes the expectation of a measurable function $f(X)$ under the process $dX = (rX - \mathcal{D}(X))d\tau + \sigma X dz$, calculated when $X(0) = S$ Define the complete transition probability $p(S,Z,\tau)$ by

$$p(S,Z,\tau) = \Pr\{X(\tau) \le Z \mid X(0) = S\} = A(S,\tau) + p^*(S,Z,\tau),$$

where $A(S,\tau)$ is the probability that the underlying process starting at $X(0) = S$ is absorbed at the origin prior to τ. Consequently, $p^*(S,Z,\tau)$ is the remaining probability, conditional on no prior absorption. Then, the put option value is given by

$$P(S,\tau) = \exp(-r\tau)\left[A(S,\tau)K + \int_0^\infty (K-Z)^+ G(S,Z,\tau)\, dZ\right],$$

where the Green function (Arrow-Debreu security) $G(S,Z,\tau) \triangleq \partial p / \partial Z = \partial p^* / \partial Z$. The first component term of the put value, which can be thought of as a bankruptcy claim, has no analog in the original Black-Scholes theory.

2.4 The Role of the Confluent Hypergeometric Functions.

We now show why the confluent hypergeometric functions will play an important role in the solution of eq.(2.1). Our general approach is similar to Titchmarsh (1962). Define $\beta = 2r/\sigma^2 \ge 0$ and $\gamma = 2D/\sigma^2 > 0$. In eq. (2.1) let $x = \gamma / S$, $t = \tau\sigma^2/2$, $f(S,\tau) = \varphi(x,t)$, and $\varphi(x,0) = \varphi_0(x)$, where $0 < x < \infty$. Then, $\varphi(x,t)$ satisfies

(2.2) $$\boldsymbol{A}\varphi \triangleq x^2\varphi_{xx} + [x^2 + (2-\beta)x]\varphi_x - \beta\varphi = \varphi_t$$

(Note: boldface always denotes a differential operator and subscripts sometimes denote differentiation with respect to the subscript variable). The Laplace transform

$$\Phi(x,s) = L[\varphi] = \int_0^{\infty} \exp(-st)\ \varphi(x,t)\, dt$$

satisfies $(\boldsymbol{A} - s)\Phi = -\varphi_0$. We solve this last equation with a Green function, which is constructed from solutions to the homogenous equation $(\boldsymbol{A} - s)\Phi_H = 0$. To invert the transform, s must be considered a complex variable throughout. Let $\Phi_H(x,s) = x^a H(x,s)$, yielding $xH_{xx} + (c+x)H_x + aH = 0$, where a and c are determined by $2a = (\beta - 1) - [(\beta + 1)^2 + 4s]^{1/2}$ and $c = 1 - [(\beta + 1)^2 + 4s]^{1/2}$. Noting the sign changes from Kummer's confluent hypergeometric equation[1], independent solutions for $H(x,s)$ are given by $M(a,c,-x)$ and $x^{1-c} M(a - c + 1, 2 - c, -x)$. We assume throughout that c is not an integer.

2.5 The Liouville Standard Form.

The Liouville standard form is a transformation of the equation $xH_{xx} + (c+x)H_x + aH = 0$ which is helpful for three purposes. First, the standard form equation has a simpler natural function space, which we use in Section 2.6 and Appendix I. Second, as discussed below, it suggests some intuition about the qualitative behavior of the solutions. Finally, it motivates a variable change from the Laplace transform variable s to an eigenvalue λ that proves convenient.

Specifically, the function $u(z,s) = x^{(c-1)/2} \exp(x/2) H(x,s)$, where $x(z) = \exp(z)$, solves the standard form $\boldsymbol{L}u \triangleq -u_{zz} + q(z)u = \lambda\, u$. Here $q(z) = \exp(2z)/4 - (\beta - 2)\exp(z)/2$ and $\lambda = -(s + \delta)$, where we introduce $\delta \triangleq (\beta + 1)^2 / 4$. In this notation $a = (\beta - 1)/2 + i\sqrt{\lambda}$ and $c = 1 + 2i\sqrt{\lambda}$. For a physical interpretation of the standard form equation (which provides some intuition about solutions), interpret $\boldsymbol{L}u = \lambda u$ as Schroedinger's equation for the probability amplitude of finding a quantum mechanical particle in the potential described by $q(z)$. Visualizing $q(z)$, one sees the potential vanishing as $z \to -\infty$, an increasingly positive barrier as $z \to \infty$, and a well of finite depth in-between (a depth controlled by β). With this type of shape, experience suggests that among the solutions will be certain bound states with a discrete spectrum, and certain oscillatory scattering states with a continuous spectrum. If β is not large enough, the potential well will be shallow and there may be no bound states. These suggestions are confirmed below in the proof of Proposition 2.1. In addition, we frequently use below probabilistic interpretations of solutions to Model I. For a discussion of the probabilistic interpretation of solutions to Schroedinger's equation, see Durrett (1984)

2.6 The Green Function and the Function Spaces.

In Section 2.4 we showed how to transform the option valuation equation into the inhomogenous equation $(\boldsymbol{A} - s)\Phi = -\varphi_0$. In this section, we discuss the solution to this equation by the Green function method. The Green function method expresses the solution to the inhomogenous equation in terms of solutions to the homogenous equation $(\boldsymbol{A} - s)\Phi = 0$. The method also leads naturally to a discussion of the relevant function spaces for the problem. In particular, the behavior of the option payoff function when the stock price is very small or very large has consequences for the qualitative behavior of Φ in the complex s-plane. Specifically, the poles of Φ in the s-plane determine the large-time behavior of the option values. In this section and the next two, we focus on payoff functions which fall off more rapidly than put and call options as $S \to 0, \infty$. In a later section, we show how to generalize these results to traditional option payoffs.

It is more convenient to use λ instead of s as our transform variable. Consider two independent solutions to $(\boldsymbol{A} + \delta + \lambda)\Phi = 0$, which we call $\xi(x,\lambda)$ and $\eta(x,\lambda)$. Then, the formal Green function solution to $(\boldsymbol{A} + \delta + \lambda)\Phi = -\varphi_0$ is given by

$$-\Phi(x,\lambda) = \eta(x,\lambda)\int_0^x \rho(y)\xi(y,\lambda)\varphi_0(y)\,dy + \xi(x,\lambda)\int_x^\infty \rho(y)\eta(y,\lambda)\varphi_0(y)\,dy, \tag{2.3}$$

where $\rho(y) = y^{-2}W^{-1}(y)$ and where W is the Wronskian of (ξ, η). Under what circumstances can we say such a solution exists? To answer this, we need some notation: for $f(x)$ a complex-valued function of a real variable x, $f \in \mathcal{L}_{2,\rho}(X,Y)$ means $\int_X^Y \rho(x)|f(x)|^2 dx < \infty$. The integrals in eq. (2.3) exist if $\xi \in \mathcal{L}_{2,\rho}(0,X)$, $\eta \in \mathcal{L}_{2,\rho}(X,\infty)$, and $\varphi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$, so these three integrability conditions are sufficient. For complex λ such that $\mathrm{Im}(\lambda) \neq 0$, suitable ξ, η in these functions spaces always exist by Weyl's limit point/limit circle theory. [For a brief review, see Weyl (1950); for a more thorough discussion, see Hille (1969)]. Equivalently, there exist solutions u_1, u_2 to $-u_{zz} + q(z)u = \lambda u$ such that $u_1 \in \mathcal{L}_2(-\infty,0)$, $u_2 \in \mathcal{L}_2(0,\infty)$, where $\mathcal{L}_2$ is the usual space of square-integrable functions with unit weight. As we shall see later, the third integrability condition (on φ_0)is not necessary for eq. (2.3) to exist, but for our first proposition 2.1 below we assume it.

The economic interpretation of the initial condition $\phi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$ can be seen by translating this condition back into S-space (stock price space), where it reads $\int_0^\infty \rho(\gamma/S)\,[f_0(S)]^2 dS < \infty$. We show below that $\rho(y) = y^{-\beta}\exp(y)$. Because $\rho(\gamma/S)$ diverges at both $S = 0$ and $S = \infty$, the contingent claim must have a payoff which vanishes at both boundaries in a prescribed manner. It is perhaps simplest to think of the initial conditions for this case as defining a function class that includes those payoff functions $f_0(S)$ which *strictly* vanish outside of any finite interval that does not include the origin (a *compact* payoff) An example of a compact payoff is the so-called butterfly spread, where one buys one call option striking at $K_1 > 0$, sells two calls striking at $K_2 > 0$, and buys one call striking at $K_3 > 0$, where $K_2 = (K_1 + K_3)/2$. A second example, further discussed following Corollary 2.1, is a barrier option with a constant (step-function) payoff in an interval.

The strongest economic effect of the condition $\phi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$ is in the valuation of very long-dated options. A very long-dated option has a value determined by the payoff function and the behavior of the stock price over long time periods. In the model here, the stock price S will typically be found near (or at) $S = 0$ or at very large values after a long time. So, if the payoff function is compact, the stock price will, with increasing probability, lie outside of the range of positive payoffs. Consequently, the option value will be very small. In fact, the exact solution will show that the value of the generalized option with payoff φ_0, where $\phi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$, will decay exponentially in time at a certain characteristic rate. In contrast, the payoff functions for traditional call options, which are not compact and which do not lie in $\mathcal{L}_{2,\rho}(0,\infty)$, retain significant value even when the option is very long-dated. So, the decay rate is zero. The payoff function for traditional put options also lies outside of $\mathcal{L}_{2,\rho}(0,\infty)$. Put option values, when the dividend rate $D > 0$, have a bankruptcy component which decays exponentially at the time value of money rate $\exp(-r\tau)$; this rate is different than the characteristic decay rate of options in the $\mathcal{L}_{2,\rho}(0,\infty)$ space. Mathematically, these different value decay rates are determined by the location of poles of Φ in the complex s (or λ) plane. And, in turn, as we shall show, the location of the poles is determined by the behavior of the payoff function φ_0 at very large and small stock prices.

2.7 The General Form of the Solution

Once we have determined $\Phi(x,\lambda)$, then the solution to eq. (2.2) is given by the usual Laplace inversion:

(2.4)
$$\varphi(x,t) = \frac{1}{2\pi i} \int_{\chi - i\infty}^{\chi + i\infty} \exp[-(\lambda + \delta)\, t]\, \Phi(x,\lambda)\, d\lambda$$

where the real number χ is to the left of any singularities in the complex λ-plane (contour C_1 in Figure 2). We will complete the contour along the arcs and contours C_2 and C_3 shown in the figure. In general, the result of evaluating eq. (2.4) via the Residue Calculus consists of contributions from both a point spectrum $\lambda_m \in \mathrm{p}(\boldsymbol{L})$, where m is an integer label, and a continuous spectrum $\lambda \in \mathrm{c}(\boldsymbol{L})$. More specifically, we can anticipate a result of the general form

$$\varphi(x,t) = \exp(-\delta t) \Big[\sum_{\lambda_m \in \mathrm{p}(\boldsymbol{L})} g_m\, \varphi_m(x) \exp(-\lambda_m t) + \int_{\lambda \in \mathrm{c}(\boldsymbol{L})} g(\lambda)\, \varphi_\lambda(x) \exp(-\lambda\, t) d\lambda \Big].$$

2.8 The Value of a General $\mathcal{L}_2$ Claim and the Value of a Call Option.

With the background discussions completed, we are ready to state the first proposition, which assumes the integrability condition $\phi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$. (Notation: we use $\theta_{(\beta > n)} = 1$ when $\beta > n$ and zero otherwise, $\Gamma(z)$ for the Gamma function, and $L_m^{(\alpha)}(x)$ for the associated Laguerre polynomials[2]. Finally, as a summation index $[x]$ denotes the greatest integer contained in x).

Proposition 2.1. *Within Model I, assume* $\int_0^\infty \rho(\gamma/S)[f_0(S)]^2\, dS < \infty$, *where the weight function* $\rho(y) = y^{-\beta}\exp(y)$. *Also define* $a_\mu = (\beta - 1)/2 + i\mu$, $c_\mu = 1 + 2i\mu$, $c_m = \beta - 2 - 2m$, *and* $d_m = c_m - 1$. *Finally, define the eigenvalues* $\lambda_m = -(\beta - 3)^2/4 + m(\beta - 3 - m)$. *Then, the solution to eq. (2.1) is given by*

(2.5) $$f(S,\tau) = \theta_{(\beta>3)} \sum_{m=0}^{[(\beta-3)/2]} g_m \eta_m(\tfrac{\gamma}{S}) \exp\left[-\tfrac{1}{2}\sigma^2(\delta+\lambda_m)\tau\right]$$

$$+\int_0^\infty g(\mu)\,\eta(\tfrac{\gamma}{S},\mu)\exp\left[-\tfrac{1}{2}\sigma^2\,(\delta+\mu^2)\tau\right]d\mu\,,$$

(2.6) *where* $$g_m = \frac{d_m}{m!\,\Gamma(\beta-2-m)} \int_0^\infty \rho(y)\,\eta_m(y)\, f_0(\frac{\gamma}{y})dy\,,$$

(2.7) $$\eta_m(x) = (-1)^m\, m!\, x^{\beta-2-m}\exp(-x) L_m^{(d_m)}(x)\,,$$

(2.8) $$g(\mu) = \mu\pi^{-2}(\sinh 2\pi\mu)\;|\Gamma(c_\mu - a_\mu)|^2 \int_0^\infty \rho(y)\,\eta(y,\mu) f_0(\frac{\gamma}{y})\,dy\,,$$

(2.9) *and* $$\eta(x,\mu) = \exp(-x)x^{a_\mu}U(c_\mu - a_\mu, c_\mu, x)\,.$$

Proof: See Appendix I.

Corollary 2.1. *Denote by* $K_\nu(x)$ *the modified Bessel function of the second kind of order ν. Then under Model I, and with the same assumptions and notation as Proposition (2.1), but in the particular case* $r=\sigma^2$ *, the solution to eq. (2.1) is given by*

(2.10) $$f(S,\tau) = \int_0^\infty g(\mu)\,\eta(\frac{\gamma}{S},\mu)\exp\left[-\sigma^2(9+4\mu^2)\,\frac{\tau}{8}\right]d\mu\,,$$

(2.11) *where* $$g(\mu) = \left(\frac{\mu\sinh 2\pi\mu}{\pi^{3/2}\cosh\pi\mu}\right)\int_0^\infty y^{-\frac{3}{2}}\exp(\tfrac{y}{2})K_{i\mu}(\tfrac{y}{2})f_0(\tfrac{\gamma}{y})\,dy\,,$$

(2.12) *and* $$\eta(x,\mu) = \sqrt{\frac{x}{\pi}}\exp(-\frac{x}{2})\,K_{i\mu}(\frac{x}{2}).$$

Proof: Since $\beta = 2$, there is no discrete spectrum. Then, the result follows from the continuous spectrum component of Proposition (2.1) and the known identities $|\Gamma(1/2+i\mu)|^2 = \pi/\cosh\pi\mu$ and $U(\nu+1/2,\ 2\nu+1,\ 2x) = \pi^{-1/2}(2x)^{-\nu}\exp(x)\,K_\nu(x)$.

An Example. Consider a *step-function payoff* where $f_0(S)=1$ for $0<S_0<S<S_1<\infty$, and vanishes otherwise. Let $\beta<3$ so there is no discrete spectrum. Using footnote 5, we find

(2.13) $$g(\mu) = |a_\mu|^{-2}(\gamma)^{-a_\mu}\left[S_1^{a_\mu}U\left(-a_\mu,c_\mu,\frac{\gamma}{S_1}\right) - S_0^{a_\mu}U\left(-a_\mu,c_\mu,\frac{\gamma}{S_0}\right)\right].$$

To obtain $f(S,\tau)$, evaluate $\int_0^\infty g(\mu)\,\eta(\frac{\gamma}{S},\mu)\exp[-\frac{1}{2}\sigma^2(\delta+\mu^2)\tau]d\mu$ numerically.

Next, we handle the problem of the call option payoff, which lies outside the scope of Proposition 2.1.

Proposition 2.2. *Within Model I, assume the call option payoff,* $f_0(S) = C_0(S) = (S-K)^+$, *where* K *is the strike price. With all other notation as in Proposition (2.1), the solution to eq. (2.1) is given by* $f(S,\tau) = C(S,\tau)$, *where*

(2.14)
$$C(S,\tau) = S - \frac{D}{r}\Big[1 - \frac{(\gamma/S)^{\beta}}{\Gamma(\beta+2)} M\big(\beta,\ \beta+2, \tfrac{-\gamma}{S}\big)\Big]$$
$$+\theta_{(\beta>1)}\,(\frac{D}{r} - K)\Big[1 - \frac{(\gamma/S)^{\beta-1}}{\Gamma(\beta)} M\big(\beta-1, \beta, \tfrac{-\gamma}{S}\big)\Big]\exp(-r\tau)$$
$$+\ \theta_{(\beta>3)}\gamma\,(\frac{\gamma}{S})^{\beta}\exp(\frac{-\gamma}{S}) \sum_{m=0}^{[(\beta-3)/2]} \xi_m(S,K)\ \exp\Big[-\tfrac{1}{2}\sigma^2(\delta+\lambda_m)\,\tau\Big]$$
$$+\ S^{(1-\beta)/2}K^{(1+\beta)/2}\exp(\frac{-\gamma}{S}) \int_0^{\infty} \xi_\mu(S,K)\ \exp\Big[-\tfrac{1}{2}\sigma^2(\delta+\mu^2)\tau\Big]d\mu\,,$$

(2.15) *where*
$$\xi_m(S,K) = \frac{(SK/\gamma^2)^{m+2}}{(m+2)!\,\Gamma(d_m)} M\big(-m-2, c_m, \tfrac{\gamma}{K}\big)\ M\big(-m, c_m, \tfrac{\gamma}{S}\big)\ ,$$

(2.16) *and*
$$\xi_\mu(S,K) \ = \ \frac{1}{2\pi}\Big|\frac{\Gamma(c_\mu - a_\mu - 2)}{\Gamma(2i\mu)}\Big|^2 (\frac{\gamma^2}{SK})^{i\mu}\, U\big(c_\mu - a_\mu - 2, c_\mu, \tfrac{\gamma}{K}\big)\ U\big(c_\mu - a_\mu, c_\mu, \tfrac{\gamma}{S}\big).$$

Proof: See Appendix II.

Two special limits of Proposition 2.2 merit discussion:

Recovery of the Merton solution as $\tau \to \infty$**.** Note that the first line of eq. (2.14) is the Merton solution[7]. The second line vanishes either because $\beta < 1$ ($r < \sigma^2/2$), or because of the factor $\exp(-r\tau)$. Hence, the second line vanishes at least as rapidly as $\exp(-\sigma^2\tau/2)$. In the third line, consider that $\lambda_m + \delta \geq \lambda_0 + \delta = 2(\beta-1) > 4$, using the restriction $\beta > 3$. Hence, the third line vanishes at least as rapidly as $\exp(-2\sigma^2\tau)$. The behavior of the fourth line, at large τ, is determined by the behavior of the integrand at small μ. It can be shown that $\xi_\mu = O(\mu^2)$ as $\mu \to 0$. In turn, this leads to the fourth line vanishing as $\exp(-\sigma^2\delta\tau/2)\tau^{-3/2} = \exp[-\sigma^2(\beta+1)^2\tau/8]\tau^{-3/2}$. Since we assume $r \geq 0$, ($\beta \geq 0$), the fourth line vanishes at least as rapidly as $\exp(-\sigma^2\tau/8)\tau^{-3/2}$.

Recovery of the Black-Scholes solution as $D \to 0$**.** In this limit $\gamma \to 0$ in eq. (2.14), which implies

$$C(S,\tau) \approx S - \theta_{(\beta>1)}K\exp(-r\tau) + S^{(1-\beta)/2}K^{(1+\beta)/2}\int_0^{\infty} \xi_\mu(S,K)\exp[-\tfrac{1}{2}\sigma^2(\delta+\mu^2)\tau]d\mu\,.$$

From the definition of *U,* as $\gamma \to 0$,

$$(\frac{\gamma}{K})^{i\mu}\,U(c-a-2, c, \frac{\gamma}{K}) \approx \frac{\Gamma(1-c)}{\Gamma(-a-1)}(\frac{\gamma}{K})^{i\mu} + \frac{\Gamma(c-1)}{\Gamma(c-a-2)}(\frac{\gamma}{K})^{-i\mu}$$

and $$(\frac{\gamma}{S})^{i\mu} U(c-a,c,\frac{\gamma}{S}) \approx \frac{\Gamma(1-c)}{\Gamma(-a+1)}(\frac{\gamma}{S})^{i\mu} + \frac{\Gamma(c-1)}{\Gamma(c-a)}(\frac{\gamma}{S})^{-i\mu} .$$

When these last two terms are multiplied together to produce the integrand $\xi_\mu(S,K)$, only the cross terms survive in the $\gamma \to 0$ limit because the factors of γ cancel. [The other terms yield integrands which behave, for small γ, as $f(\mu)\exp(\pm 2i\mu \ln\gamma)$. Such integrands are increasingly oscillatory as $\gamma \to 0$, yielding contributions to $C(S,\tau)$ which vanish as $O(1/|\ln\gamma|)$]. In summary, and after some simplification, we have the leading behavior as $\gamma \to 0$,

$$\xi_\mu(S,K) \approx \frac{1}{2\pi}[\exp(i\mu z)\frac{1}{(-\frac{\beta}{2}+\frac{1}{2}-i\mu)(-\frac{\beta}{2}-\frac{1}{2}-i\mu)} + c.c.],$$

using $z = \ln(K/S)$ and *c.c.* = complex conjugate. Using $\xi_\mu(S,K) = \xi_{-\mu}(S,K)$ and making the change of integration variable to $\nu = \mu - iz/(\sigma^2\tau)$ yields

$$\int_0^\infty \xi_\mu(S,K)\, \exp[-\tfrac{1}{2}\sigma^2(\delta+\mu^2)\tau]d\mu = \frac{1}{2}\int_{-\infty}^\infty \xi_\mu(S,K)\, \exp[-\tfrac{1}{2}\sigma^2(\delta+\mu^2)\tau]d\mu$$

$$= \frac{1}{4\pi}\exp(-\frac{1}{2}\sigma^2\delta\tau - \frac{z^2}{2\sigma^2\tau}) \int_{-iz/(\sigma^2\tau)-\infty}^{-iz/(\sigma^2\tau)+\infty} [\exp(-\frac{1}{2}\sigma^2\tau\nu^2)\Big(\frac{i}{\nu+ic_1} - \frac{i}{\nu+ic_2}\Big) + c.c.]\, d\nu ,$$

using

$$c_1 = \frac{z}{\sigma^2\tau} - \frac{1}{2}(\beta+1) \quad \text{and} \quad c_2 = \frac{z}{\sigma^2\tau} - \frac{1}{2}(\beta-1) .$$

The integral may be evaluated by completing the integration contour as a rectangle in the complex ν-plane, where one side is along the real ν axis, and using the residue theorem. The integrations along the sides of the rectangle parallel to the imaginary axis vanish as these sides are extended to infinity. From the residue theorem, there are potential residue contributions at the poles $\nu = -ic_{1,2}$. Whether or not the rectangle encloses a pole depends on the sign of $c_{1,2}$ and the magnitude of β. The cases are as follows: (I) the pole at $\nu = -ic_1$ is enclosed if $c_1 > 0$ and any $\beta \ge 0$. If $c_1 < 0$ this first pole is not enclosed. (II) the pole at $\nu = -ic_2$ is enclosed if $c_2 > 0$ and $\beta > 1$. Otherwise this second pole is not enclosed. In addition, the integrals along the real ν-axis are evaluated using the formula, for *a* real and positive and *c* real and non-zero:

$$\int_{-\infty}^\infty \frac{\exp(-a\nu^2)}{\nu+ic}d\nu = -2\pi i\,(\text{sign } c)\exp(ac^2)\Phi(-|c|\sqrt{2a}) ,$$

where $\Phi(x)$ is the cumulative normal function[8]. The result is

$$S^{(1-\beta)/2}K^{(1+\beta)/2}\int_0^\infty \xi_\mu(S,K)\, \exp[-\tfrac{1}{2}\sigma^2(\delta+\mu^2)\tau]d\mu \approx S[-1+\Phi(d_1)] + \exp(-r\tau)K[\theta_{(\beta>1)} - \Phi(d_2)]$$

using $-\sigma\tau c_1 = d_1$ and $-\sigma\tau c_2 = d_2$. Hence $C(S,\tau) \approx S\Phi(d_1) - K\exp(-r\tau)\Phi(d_2)$, which is the Black-Scholes formula. ■

Economic Interpretation, and Examples.

Numerical computations were performed in the Mathematica system. In that system, the confluent hypergeometric functions $M(a,c,x)$ and $U(a,c,x)$ (for complex parameters) are built-in functions and were evaluated at the (default) machine precision. The branch cut integrations, such as the $d\mu$ integration in Proposition 2.2, were performed with the built-in numerical integration function. The integrals were evaluated from $\mu = 0$ to a cut-off value for μ which was increased until no significant changes were observed. In general, these integrations take longer to perform as the time τ is reduced because the integrands become more oscillatory and the cut-offs must be increased until the exponential damping term $\exp(-\sigma^2\mu^2\tau/2)$ become effective.

The most dramatic difference between the continuous dollar dividend model and the continuous yield model is the asymptotic behavior as $\tau \to \infty$. For example, using some typical parameter values, these two model values are plotted in Figure 1. As suggested by the figure, and already evident from the Merton partial solution, the continuous dollar dividend model tends asymptotically to a positive value. In contrast, the constant yield model value tends to zero. This can be simply understood by thinking of the call value as the discounted expectation

(2.17) $$\begin{aligned} C(S,\tau) &= \exp(-r\tau)E_S[(X_\tau - K)^+] = \exp(-r\tau)E_S[(X_\tau - K) + (K - X_\tau)^+] \\ &= F(S,\tau) - \exp(-r\tau)K + P(S,\tau), \end{aligned}$$

which is put-call parity. Note that $F(S,\tau) = \exp(-r\tau)E_S[X_\tau]$ is the fair value of a (modified) forward. A modified forward is the non-dividend paying claim with the payoff function $F_0(S_{T^*}) = S_{T^*}$, where S_{T^*} is the terminal stock price. As $\tau \to \infty$, the put value vanishes, since

$$0 \le P(S,\tau) = \exp(-r\tau)\, E_S[(K - X_\tau)^+] \le K\, \exp(-r\tau).$$

Consequently, as $\tau \to \infty$, $C(S,\tau) \approx F(S,\tau) + O(\exp(-r\tau))$. Purchasing the stock today can be thought of as a claim on all the dividends paid up to an arbitrary time *T* plus the value of the stock at that time S_T, which leads to $F(S,\tau) = S - E_S[\int_0^\tau \exp(-ru)\, \mathcal{D}(X_u)\, du]$. As $\tau \to \infty$, the second term is simply the discounted expected value of the dividend stream. To summarize, for an *arbitrary* time homogeneous dividend process:

(2.18) $$C(S,\tau) \underset{\tau\to\infty}{\approx} S - E_S\left[\int_0^\tau \exp(-ru)\, \mathcal{D}(X_u)\, du\right] + O(\exp(-r\tau)).$$

In words, the call option value tends asymptotically to the stock price less the present value (Laplace transform) of the dividend stream. Now let us apply this to the two cases under consideration.

First, in the case of the constant dollar dividend rate, since $0 \le \mathcal{D}(X_u) \le D$, the dividend stream value is positive and bounded below by zero and above by D/r. Consequently, we see that, as $\tau \to \infty$, $C(S,\tau) \ge S - D/r$. In contrast, in the alternative constant *yield* model, where $\mathcal{D}(X_u) = \delta X_u$, it is well-known that $E_S[X_u] = S\exp[(r-\delta)u]$, which leads to $C(S,\tau) \approx F(S,\tau) = S\exp(-\delta\tau) + O(\exp(-r\tau)) \to 0$.

Since the two models have the same initial conditions, they obviously must converge in value at the other limit $\tau \to 0$. Table I shows the two model option values for some typical parameter values. Because of the noted large-time behavior, there is a general tendency (which is evident from Table I) for the constant dividend model values to be greater than their comparable constant yield model values. However, this is not always the case, which is illustrated in Table II , which reports difference values only and shows some negative entries. Table II can be understood from the idea that large strike prices favor the constant dividend model, since the dividend payments are relatively less for large increases in the stock price. Conversely, smaller strikes work against the constant dividend model, because of the relatively larger dividend payouts near the strike price and the increasing possibility of bankruptcy.

2.9 The Bankruptcy Claim and the Absorption Kernel.

The unit bankruptcy claim is defined to be a European-style claim paying \$1 on expiration if the stock price has been absorbed at zero and otherwise paying nothing. Let $B(S,\tau)$ denote the fair value of the claim with τ periods to expiration when the stock price is S. Then, because the claim can be expressed as the discounted expected value of its payoff cash flows, we can calculate it from $B(S,\tau) = \exp(-r\tau)A(S,\tau)$, where $A(S,\tau)$ is the probability of absorption by expiration. Explicitly,

$$(2.19) \qquad A(S,\tau) \triangleq \Pr\{X(t_o) = 0,\ t_o \le \tau \mid X(0) = S\},$$

where t_o is the moment when the risk-adjusted process $dX = (rX - \mathcal{D}(X))d\tau + \sigma X dz$ first reaches the origin, starting from $X(0) = S$.

Proposition 2.3. *Let* $A(S,\tau)$ *be the absorption probability defined by eq. (2.19) and let* $A_\infty(S)$ *be the probability of ultimate absorption. Define* $\eta(x,\mu)$, a_μ, c_μ, c_m, *and* d_m *as in Proposition 2.1 and denote the incomplete Gamma function by* $\Gamma(\beta,x)$. *Then,* $A(S,\tau)$ *and* $A_\infty(S)$ *are related by*

$$(2.20) \qquad A(S,\tau) = A_\infty(S) + A_p(S,\tau) + A_c(S,\tau),$$

$$(2.21) \quad \textit{where} \qquad A_\infty(S) = \begin{cases} 1 - \dfrac{\Gamma(\beta-1, \frac{\gamma}{S})}{\Gamma(\beta-1)} & r > \sigma^2/2 \\ 1 & r < \sigma^2/2 \end{cases},$$

(2.22)

$$A_p(S,\tau) = \theta_{(\beta>3)} \exp(\frac{-\gamma}{S}) \sum_{m=0}^{[(\beta-3)/2]} \frac{(-1)^{m+1}}{m!} \frac{(\frac{\gamma}{S})^{\beta-2-m}}{\Gamma(d_m)} \frac{\exp\{-\frac{1}{2}[\beta-2+m(\beta-3-m)]\,\sigma^2\tau\}}{[\beta-2+m(\beta-3-m)]} M(-m,c_m,\frac{\gamma}{S}),$$

(2.23) *and*
$$A_c(S,\tau) = -\int_0^\infty \bar{g}(\mu)\,\eta(\frac{\gamma}{S},\mu)\exp\left\{-\frac{1}{2}\left[\mu^2+\frac{(\beta-1)^2}{4}\right]\sigma^2\tau\right\}d\mu\,,$$

(2.24) *using*
$$\bar{g}(\mu) = \frac{1}{2\pi}\left|\frac{\Gamma(c_\mu-a_\mu)}{\Gamma(2i\mu)}\right|^2 \frac{1}{\mu^2+\frac{(\beta-1)^2}{4}}\;.$$

Proof: It is known (see, for example, Karlin and Taylor (1981)) that $A(S,\tau)$ satisfies the backward Kolmogorov equation

(2.25)
$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 A}{\partial S^2} + (rS-D)\frac{\partial A}{\partial S} = \frac{\partial A}{\partial \tau},$$

subject to $A(S=0,\tau)=1$, $A(S=\infty,\tau)=0$, and $A(S,\tau=0)=0,\ (S>0)$. Introduce the same reduced variables as before: $x=\gamma/S$ and $t=\sigma^2\tau/2$ and let $A(S,\tau)=\tilde{A}(x,t)$. Define the *absorption kernel* to be the Laplace transform of the absorption probability with respect to the reduced time variable *t;* that is

$$K_{abs}(x,s) = \int_0^\infty \exp(-st)\tilde{A}(x,t)\,dt\,.$$

With our same techniques, we find that $K_{abs}(x,s)=x^{\bar{a}}H(x,s)$, where $xH_{xx}+(\bar{c}+x)H_x+\bar{a}H=0$. The new constants are determined by $2\bar{a}=(\beta-1)-[(\beta-1)^2+4s]^{1/2}$ and $\bar{c}=2\bar{a}+2-\beta$. Introducing $\bar{\lambda}=-[s+(\beta-1)^2/4]$, then in this notation $\bar{a}=(\beta-1)/2+i\sqrt{\bar{\lambda}}$ and $\bar{c}=1+2i\sqrt{\bar{\lambda}}$. We again place the branch cut in the complex $\bar{\lambda}$-plane along the positive real axis and define $\operatorname{Im}\sqrt{\bar{\lambda}}>0$ throughout the cut plane. Applying the boundary conditions, we find

$$K_{abs}(x,s) = \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})}\,\frac{1}{s}\,x^{\bar{a}+1-\bar{c}}M(\bar{a}+1-\bar{c},\ 2-\bar{c},-x)\,.$$

The inversion contour in the $\bar{\lambda}$ - plane must lie to the left of the various poles. The controlling (left-most) case is the pole at $s=0$; that is, $\bar{\lambda}^*=-(\beta-1)^2/4$. The residue at this pole is given by

$$\text{Residue at the pole } \bar{\lambda}^* = \begin{cases} \dfrac{(-1)}{\Gamma(\beta)}x^{\beta-1}M(\beta-1,\beta,-x) & (\beta>1) \\ (-1) & (\beta<1) \end{cases}$$

In addition, there are the Gamma function poles $\bar{\lambda}_m$ determined by $1-\bar{a}_m=-m$ or, equivalently

$$\bar{\lambda}_m = -\frac{(\beta-3)^2}{4}+m(\beta-3-m) \qquad m=0,\ 1,\ \ldots\ ,\ [\tfrac{(\beta-3)}{2}].$$

and occur only if $\beta>3$. Consequently, in performing the Laplace inversion using the contours in Figure 2, there are three contributions. The contribution from the first pole yields the probability of ultimate absorption

$A_\infty(S)$. The contribution from the Gamma function poles yields $A_p(S,\tau)$. The branch cut contribution yields $A_c(S,\tau)$. The explicit formulas, given above, are obtained by straight-forward computations.[9] ■

Economic Interpretation. The two cases of eq. (2.21) for the ultimate absorption probability $A_\infty(S)$ represent parameter regimes where ultimate bankruptcy is either certain or merely possible. This can be understood by considering the growth rate process for the stock price, which is given by $Y = \ln S$. From Ito's lemma, $dY = \mu(Y)dt + \sigma\, dz$, where $\mu(Y) = r - \sigma^2/2 - D\exp(-Y)$. If $r < \sigma^2/2$, the growth rate is always negative which leads to ultimate bankruptcy with probability 1. On the other hand, if $r > \sigma^2/2$, the growth rate can be positive for large enough stock prices, and this allows the possibility of an ultimate escape to arbitrarily large stock prices.

2.10 The Bankruptcy Claim and the Green Function.

Define the transition probability $p(S,Z,\tau)$ by

(2.26) $$p(S,Z,\tau) = \Pr\{X(\tau) \le Z \mid X(0) = S\}\ .$$

It also satisfies the backward Kolmogorov equation (2.25), but with the boundary conditions

$$p(S=0,Z,\tau)=1\ ,\ p(S=\infty,Z,\tau)=0\ \text{, and }\ p(S,Z,\tau=0)=\theta_{(S\le Z)}\ .$$

Note that we can write

(2.27) $$p(S,Z,\tau) = \Pr\{X(\tau)=0 \mid X(0)=S\} \ + \ \Pr\{0 < X(\tau) \le Z \mid X(0)=S\}$$

$$\triangleq A(S,\tau) + p^{*}(S,Z,\tau)\ .$$

The second term in eq. (2.27) is the probability of the stock price being less than or equal to Z conditional on no absorption by τ. The first term is given by Proposition 2.3. Consider the transition density $G(S,Z,\tau) \triangleq \partial p / \partial Z = \partial p^{*} / \partial Z$. The Green function $G(S,Z,\tau)$ is a solution to the backward Kolmogorov eq. (2.25) with the boundary conditions $G(S=0,Z,\tau)=0$, $G(S=\infty,Z,\tau)=0$, and $G(S,Z,\tau=0)=\delta(Z-S)$, where $\delta(\cdot)$ is the Dirac delta function.

Interpretation. The function $\exp(-r\tau)\ G(S,Z,\tau)$ has the economic interpretation of an *Arrow-Debreu* security for Model I. That is, it represents the value of a contingent claim that pays nothing unless the particular state $S=Z>0$ occurs upon expiration of the claim. As one sees from its construction, one must carefully integrate $G(S,Z,\tau)$ to obtain $p(S,Z,\tau)$; the integration constant is important because it is also the absorption probability. Equivalently, in constructing the value of a general claim from the Arrow-Debreu claim, the integration constant represents the value of the bankruptcy claim. This idea is developed explicitly in our discussion in Section 2.11 on the value of the put option.

Proposition 2.4. *The function* $G(S,Z,\tau) = \ \gamma^{-1} f(\gamma/S, \gamma/Z, \sigma^2\tau/2)$, *where*

(2.28) $$f(x,y,t) = \theta_{(\beta>3)}\, x^{\beta-2}\exp(-x) \sum_{m=0}^{[(\beta-3)/2]} \frac{m!}{\Gamma(d_m)}\,(xy)^{-m}\, L_m^{(d_m)}(x)\, L_m^{(d_m)}(y)\exp\left\{-\left[\lambda_m + \frac{(\beta-1)^2}{4}\right]t\right\}$$

$$+x^{\beta-2}\exp(-x)\int_0^{\infty} g(\mu)(xy)^{c_\mu-a_\mu}\ U(c_\mu-a_\mu,c_\mu,x)\ U(c_\mu-a_\mu,c_\mu,y)\ \exp\left\{-\left[\mu^2+\frac{(\beta-1)^2}{4}\right]t\right\}d\mu\ ,$$

(2.29) *using* $$g(\mu) = \frac{1}{2\pi}\left|\frac{\Gamma(c_\mu-a_\mu)}{\Gamma(2i\mu)}\right|^2 .$$

Proof: Since $G(S,Z,\tau=0)=f_0(S)=\delta(Z-S)$, the integrability assumptions of Proposition 2.1 are satisfied. Moreover, eq.(2.5) yields a solution to eq. (2.1) that must vanish at both $S=0$ and $S=\infty$ because of the integrability condition. In addition, it can be interpreted as $\exp(-r\tau)$ times a solution to the backward Kolmogorov eq.(2.25). Hence, this particular solution must be $\exp(-r\tau)G(S,Z,\tau)$, with the explicit formulas given above. ■

The Conservation of Probability. After a finite time, the risk-adjusted process $dX = (rX - \mathcal{D}(X))d\tau + \sigma X dz$ beginning at $X(0) = S$ must either have been absorbed at the origin or be found at some $Z > 0$. This suggests the following proposition:

Proposition 2.5 (Conservation of probability). *With the absorption probability* $A(S,\tau)$ *given by Proposition 2.3 and the Green function* $G(S,Z,\tau)$ *given by Proposition 2.4, we have*

(2.30) $$1 = A(S,\tau) + \int_0^\infty G(S,Z,\tau)dZ .$$

Proof: See Appendix III.

Additional Remarks.

(i) To verify that, indeed, $\lim_{D\to 0} G(S,Z,\tau)$ is given by the lognormal density, let $\gamma \to 0$ in eq. (2.28). In this limit the first line with the discrete sum vanishes, and in the second line

$$g(\mu)(xy)^{i\mu} U(c_\mu - a_\mu, c_\mu, x) U(c_\mu - a_\mu, c_\mu, y) \approx \frac{1}{\pi}\cos(\mu\bar{z}) ,$$

using $\bar{z} = \ln(Z/S)$. Then, using

$$\int_0^\infty \cos(\mu\bar{z}) \exp(-\mu^2 t) d\mu = \frac{1}{2}\sqrt{\frac{\pi}{t}} \exp(-\frac{\bar{z}^2}{4t}) ,$$

we find $$G(S,Z,\tau) = \frac{1}{Z\sqrt{2\pi\sigma^2\tau}} \exp\left[-(\beta-1)^2\sigma^2\frac{\tau}{8} + (\beta-1)\frac{\bar{z}}{2} - \frac{\bar{z}^2}{2\sigma^2\tau}\right]$$

$$= \frac{1}{Z\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{(\bar{z}-\mu\tau)^2}{2\sigma^2\tau}\right], \quad \text{where} \quad \mu = r - \sigma^2/2 ,$$

which is the lognormal density.

(ii) Note that $G(S,Z,\tau)$ also satisfies the forward Kolmogorov (or Fokker-Planck) equation with respect to its second argument:

$$\frac{1}{2}\sigma^2 (Z^2 G)_{ZZ} - [(rZ - D)\, G]_Z = G_\tau .$$

(iii) Technically, $G(S,Z,\tau)$ was only defined for $Z > 0$ because $p(S,Z,\tau)$ was only defined for $Z > 0$. However, in fact: $0 < \lim_{Z\to 0} G(S,Z,\tau) < \infty$.

2.11 The Value of the Put Option.

Given our calculations so far, there are at least three ways to compute the fair value for a put option: (I) the discounted expected value method, which is an integration from the complete transition probability, or (II) using put-call parity, or (III) a direct calculation from the differential equation analogous to the call option calculation We will report results from both methods (I) and (II) since they lead to formulas that look different although, in fact, they agree with each other.

Method (I): The discounted expected value.

Like most European-style contingent claims, the put option value can be thought of as the discounted expected value of its payoff function (assuming this expectation exists). In terms of the absorption probability and the transition density given in the previous section, this interpretation is expressed by $P(S,\tau) = P_1(S,\tau) + P_2(S,\tau)$, where

(2.31) $$P_1(S,\tau) = \exp(-r\tau)\, A(S,\tau) K, \quad \text{and} \quad P_2(S,\tau) = \exp(-r\tau) \int_0^\infty (K-Z)^+ \, G(S,Z,\tau)\, dZ.$$

In words, this means that the put option is worth a bankruptcy claim plus its discounted expected terminal value, the latter conditional on no absorption or bankruptcy. The bankruptcy claim (part 1 value) is reported in Proposition 2.3.

Proposition 2.6 *The function* $P_2(S,\tau) = \gamma f_2(\gamma/S, \gamma/K, \sigma^2\tau/2)$, *where*

(2.32) $$f_2(x,k,t) = \theta_{(\beta>3)} x^{\beta-2} \exp(-x) \sum_{m=0}^{[(\beta-3)/2]} \frac{x^{-m}}{\Gamma(d_m)} L_m^{(d_m)}(x) I_m(k) \exp\left\{-\left[\lambda_m + \frac{(\beta+1)^2}{4}\right] t\right\}$$
$$+ x^{\beta-2} \exp(-x) \int_0^\infty g(k,\mu)\, x^{c_\mu - a_\mu}\, U(c_\mu - a_\mu, c_\mu, x) \exp\left\{-\left[\mu^2 + \frac{(\beta+1)^2}{4}\right] t\right\} d\mu,$$

(2.33) *using* $$I_m(k) = \sum_{j=0}^{m} \frac{(-m)_j}{(m+2-j)(m+1-j)} \frac{(c_m)_m}{(c_m)_j} \frac{k^{j-m-2}}{j!},$$

(2.34) *and* $$g(k,\mu) = \frac{1}{\pi} \left| \frac{\Gamma(c_\mu - a_\mu)}{\Gamma(2i\mu)} \right|^2 \left\{ \frac{1 - k^{c_{\bar\mu} - a_{\bar\mu} - 2}\, U(c_{\bar\mu} - a_{\bar\mu} - 2, c_\mu, k)}{\left[\mu^2 + \frac{(\beta-1)^2}{4}\right]\left[\mu^2 + \frac{(\beta+1)^2}{4}\right]} - \frac{k^{-1}}{\left[\mu^2 + \frac{(\beta-1)^2}{4}\right]} \right\}.$$

Proof: Compute $P_2(S,\tau) = \exp(-r\tau) \int_0^\infty (K-Z)^+ G(S,Z,\tau)\, dZ$, using $G(S,Z,\tau)$ from Proposition 2.4 and the integration formulas given in the footnotes.

Method (II): put-call parity. Use the put-call parity eq. (2.17) and the following Proposition:

Proposition 2.7 *The value of the modified forward* $F(S,\tau)$ *is given by*

(2.35)
$$F(S,\tau) = S - \frac{D}{r}\Big[1 - \frac{(\gamma/S)^{\beta}}{\Gamma(\beta+2)} M\Big(\beta,\ \beta+2, \frac{-\gamma}{S}\Big)\Big]$$
$$+\theta_{(\beta>1)}\ \frac{D}{r}\Big[1 - \frac{(\gamma/S)^{\beta-1}}{\Gamma(\beta)} M\Big(\beta-1, \beta, \frac{-\gamma}{S}\Big)\Big]\exp(-r\tau)$$
$$+\ \theta_{(\beta>3)}\gamma\ (\frac{\gamma}{S})^{\beta}\exp(\frac{-\gamma}{S}) \sum_{m=0}^{[(\beta-3)/2]} \xi_m(S,0)\ \ \exp\Big[-\frac{1}{2}\sigma^2(\delta+\lambda_m)\,\tau\Big]$$
$$+\ S\,(\frac{\gamma}{S})^{(\beta+1)/2}\ \exp(\frac{-\gamma}{S}) \int_0^{\infty} h_u(S)\ \exp\Big[-\frac{1}{2}\sigma^2(\delta+\mu^2)\tau\Big]d\mu\ ,$$

(2.36) *where*
$$\xi_m(S,0) \ =\ \frac{(-S/\gamma)^{m+2}}{(m+2)!\Gamma(d_m)(c_m)_{m+2}} M\Big(-m, c_m, \frac{\gamma}{S}\Big)$$

(2.37) *and*
$$h_u(S) = \frac{1}{2\pi}\left|\frac{\Gamma(c_\mu - a_\mu - 2)}{\Gamma(2i\mu)}\right|^2 (\frac{\gamma}{S})^{i\mu}\, U(c_\mu - a_\mu, c_\mu, \frac{\gamma}{S})\,.$$

Proof: Use $F(S,\tau) = \lim_{K\to 0} C(S,\tau)$, where the limit is taken in the Proposition 2.2 result for $C(S,\tau)$.

Discussion. Let us first review why the two put formulas must be identical. First, define

$$I(S,K,\tau) \triangleq \exp(-r\tau)\int_K^{\infty}(Z-K)G(S,Z,\tau)\ dZ$$

Then, rearrangement of the integration limits yields

$$P_1(S,\tau)+P_2(S,\tau)\ = \exp(-r\tau)\ [A(S,\tau)K + \int_0^{\infty}(K-Z)^+ G(S,Z,\tau)\ dZ]$$
$$= \exp(-r\tau)K[A(S,\tau) + \int_0^{\infty} G(S,Z,\tau)\ dZ] + [I(S,K,\tau) - I(S,0,\tau)]\,.$$

But $A(S,\tau) + \int_0^{\infty} G(S,Z,\tau)\ dZ = 1$ by Proposition 2.5. We note that (i) $G(S,Z,\tau)$ is a solution to the option valuation eq. (2.1), (ii) $G(S,K,\tau=0) = \delta(S-K)$; hence $I(S,K,\tau=0) = (S-K)^+$, (iii) $I(S,K,\tau) < \infty$ because it was computed explicitly in Appendix II. Hence, we conclude that $I(S,K,\tau) = C(S,K,\tau)$, or in summary, $P(S,\tau) = P_1(S,\tau) + P_2(S,\tau) \ = \exp(-r\tau)K + C(S,K,\tau) - C(S,0,\tau)$, which is the assertion.

Table III gives numerical examples; we show the breakdown of the total value into the two parts of Method I and we have also verified that the Method II formula produces the same numerical values.

3. THE TERM STRUCTURE OF INTEREST RATES

3.1. Problem Definition.

In this section, we discuss the valuation of arbitrage-free bond, bond option, and other generalized contingent claim prices in a stationary, single factor, interest rate term structure model. In this theory, we can continue to use notation from the earlier section but with these interpretations: now r_T becomes the non-negative random interest rate at time *T.* A (non-coupon paying) bond or other generalized claim expires or matures at time T^*, at which time it has the specified payoff $f_0(r_{T^*})$. Model II is closely related to Merton's (1975) process for the short-term interest rate $dr = (ar - br^2)dT + \sigma r\, dz$, which he deduced from an economic growth model. Under the further assumption that the market price of risk is a constant, then arbitrage arguments show that the value of the claim $f(r,\tau)$ again depends only upon r_T and $\tau = T^* - T$. The claim satisfies (Model II):

$$\tag{3.1} \tfrac{1}{2}\sigma^2 r^2 \frac{\partial^2 f}{\partial r^2} + (Ar - Br^2)\frac{\partial f}{\partial r} - rf = \frac{\partial f}{\partial \tau},$$

where $\tau > 0,\ 0 \le r < \infty$, and $A > 0, B > 0$ and $\sigma > 0$ are constants independent of *r*. For the pure discount bond $f_0(r_{T^*}) = 1$, but we handle general payoff functions, again subject to certain integrability restrictions. Depending upon the parameter values, the risk-adjusted interest rate process underlying eq. (3.1) may have a steady-state probability distribution; when it does, we call this the equilibrium case. When no such distribution exists, we call this the non-equilibrium case. Our solutions provide the dynamics for both cases. Model II has the virtue that percentage changes in rates have constant variance, which is perhaps the *natural* extension of eq.(2.1) to fixed income securities. Note that if one wants to value a coupon bond with continuous payment of *D* dollars per year*,* the coupon bond solution $f(r,\tau;D)$ is determined by an integration:

$$f(r,\tau;D) = f(r,\tau) + D\int_0^{\tau} f(r,s)\, ds.$$

Boundary Conditions. Both the origin and infinity for *r* are singular points, and hence do not necessarily admit the specification of boundary values. The underlying risk-neutral process is $dX = (AX - BX^2)d\tau + \sigma X dz$, starting at $X(0) = r$. Since we assume *A*, *B*, and σ positive, then by Feller's criteria, both the origin and infinity are natural boundaries. Hence, solutions to eq. (3.1) are determined uniquely by the initial conditions (payoff functions) and no other boundary conditions may be specified. As in Model I, we begin with a general payoff function that is square-integrable with respect to a weight function and then subsequently consider the particular payoff function associated with the discount bond.

Proposition 3.1. *Within Model II, assume* $\int_0^\infty \rho(\gamma r)[f_0(r)]^2 dr < \infty$, *where the weight function* $\rho(y) = y^{\beta-2}\exp(-y)$, *using* $\beta = 2A/\sigma^2$, *and* $\gamma = 2B/\sigma^2$. *Also define* $\alpha = 1/B$, $\omega = (\beta-1)^2/4$, $a_\mu = \alpha - (\beta-1)/2 + i\mu$, $c_\mu = 1 + 2i\mu$, *and* $c_m = \beta - 2\alpha - 2m$. *Finally, define eigenvalues* $\lambda_m = -(\beta - 1 - 2\alpha - 2m)^2/4$. *Then, the solution to eq. (3.1) is given by*

$$f(r,\tau) = \left\{ \theta_{(\beta>1+2\alpha)} \sum_{m=0}^{[(\beta-1)/2-\alpha]} g_m\, \eta_m(\gamma r) \exp\left[-\tfrac{1}{2}\sigma^2(\omega + \lambda_m)\tau\right] + \int_0^\infty g(\mu)\eta(\gamma r,\mu) \exp\left[-\tfrac{1}{2}\sigma^2(\omega + \mu^2)\tau\right] d\mu \right\}, \tag{3.2}$$

(3.3) *where*
$$g_m = \frac{(c_m - 1)}{m!\,\Gamma(\beta - 2\alpha - m)} \int_0^\infty \rho(y)\eta_m(y) f_0(\gamma y)\, dy, \tag{3.3}$$

$$\eta_m(x) = \frac{\Gamma(1-c_m)}{\Gamma(1+2\alpha-\beta+m)} x^{-(\alpha+m)} M(-m, c_m, x), \tag{3.4}$$

$$g(\mu) = \frac{1}{2\pi}\left|\frac{\Gamma(a_\mu)}{\Gamma(2i\mu)}\right|^2 \int_0^\infty \rho(y)\,\eta(y,\mu) f_0(\gamma y) dy, \tag{3.5}$$

(3.6) *and*
$$\eta(x,\mu) = x^{a_\mu - \alpha} U(a_\mu, c_\mu, x). \tag{3.6}$$

Proof (*Sketch*)*:* The model is similar to Model I after substitutions, so we just indicate the differences. In eq.(3.1), let $x = 2Br/\sigma^2$, $t = \tau\sigma^2/2$, $f(r,\tau) = \varphi(x,t)$, and $\varphi(x,0) = \varphi_0(x)$. Then $\varphi(x,t)$ satisfies

$$\boldsymbol{B}\varphi \triangleq x^2\varphi_{xx} + (\beta x - x^2)\varphi_x - \alpha x\varphi = \varphi_t, \tag{3.7}$$

where $t > 0$ and $0 < x < \infty$. Equivalently, the Laplace transform $\Phi(x,s)$ satisfies $(\boldsymbol{B} - s)\Phi = -\varphi_0$. Substitute $\Phi_H(x,s) = x^\kappa H(x,s)$, yielding $xH_{xx} + (c - x)H_x - aH = 0$, where $a = \alpha + \kappa$, $c = \beta + 2\kappa$, and $\kappa = -(\beta-1)/2 + [s+\omega]^{1/2}$. Note the sign changes from Model I.

The substitution $u(z,s) = \exp(-x/2)x^{(c-1)/2}H(x,s)$, where $x(z) = \exp(z)$, again yields $\boldsymbol{L}u = -u_{zz} + q(z)u = \lambda u$. But now we have $q(z) = \exp(2z)/4 + (\alpha - \beta/2)\exp(z)$ and $\lambda = -(s+\omega)$. In this notation $a = \alpha - (\beta-1)/2 + i\sqrt{\lambda}$, $c = 1 + 2i\sqrt{\lambda}$, and $\kappa = -(\beta-1)/2 + i\sqrt{\lambda}$.

Appropriate solutions $\xi \in \mathcal{L}_{2,\rho}(0,X)$ and $\eta \in \mathcal{L}_{2,\rho}(X,\infty)$ to $(\boldsymbol{B} + \omega + \lambda)\Phi = 0$ are

$$\xi(x,\lambda) = x^\kappa w_2(x) = x^{\kappa+1-c} M(a - c + 1, 2 - c, x),$$

and
$$\eta(x,\lambda) = x^\kappa w_5(x) = x^\kappa U(a,c,x).$$

The Wronskian of these two solutions is $K(\lambda)\exp(x)\,x^{-\beta}$, and hence $\rho(x)=\exp(-x)x^{\beta-2}/K(\lambda)$, where $K(\lambda)=-\Gamma(2-c)/\Gamma(a-c+1)$. Bound states exist only when $\beta>1+2\alpha$ and the eigenvalues are given by $\lambda_m=-(\beta-1-2\alpha-2m)^2/4$ for $m=0,1,\ldots,[(\beta-1)/2-\alpha]$. We find $\xi_m=k_m\eta_m$, but now $k_m=\Gamma(1+2\alpha-\beta+m)/\Gamma(1+2\alpha-\beta+2m)$ and $K'(\lambda_m)=(-1)^{m+1}m!\,\Gamma(-2\alpha+\beta-2m)/(1+2\alpha-\beta+2m)$. Analogous calculations to those in the Appendix I proof of Proposition (2.1) again yield a point spectrum and continuous spectrum contribution $\varphi=\varphi_p+\varphi_c$, where now

(3.8) $$\varphi_p(x,t)=\theta_{(\beta>1+2\alpha)}\sum_{m=0}^{[(\beta-1)/2-\alpha]} g_m\,\eta_m(x)\,\exp[-(\omega+\lambda_m)t],$$

(3.9) $$\varphi_c(x,t)=\int_0^\infty g(\lambda)\eta(x,\lambda)\exp[-(\omega+\lambda)t]\,d\lambda\,,$$

(3.10) $$g_m=\frac{(\beta-2\alpha-1-2m)}{m!\,\Gamma(\beta-2\alpha-m)}\int_0^\infty y^{\beta-2}\exp(-y)\,\eta_m(y)\,\phi_0(y)\,dy\,,$$

(3.11) $$\eta_m(x)=\frac{\Gamma(1+2\alpha-\beta+2m)}{\Gamma(1+2\alpha-\beta+m)}x^{-(\alpha+m)}\,M(-m,\beta-2\alpha-2m,x)\,,$$

(3.12) and $$g(\lambda)=\frac{1}{4\pi\sqrt{\lambda}}\left|\frac{\Gamma[(1-\beta)/2+\alpha+i\sqrt{\lambda}]}{\Gamma(2i\sqrt{\lambda})}\right|^2\int_0^\infty y^{\beta-2}\exp(-y)\,\eta(y,\lambda)\,\phi_0(y)\,dy\,.$$

Changing integration variable to $\lambda=\mu^2$ yields the proposition as stated. ■

Proposition 3.2. *Within Model II, assume the discount bond payoff* $f_0(r)=B_0(r)=1$ *and the other notation from Proposition 3.1 above. Then, the solution to eq. (3.1) is given by*

(3.13) $$B(r,\tau)=\theta_{(\beta\le 1)}\frac{\Gamma(1+\alpha-\beta)}{\Gamma(1-\beta)}U(\alpha,\beta,\gamma r)+f(r,\tau),$$

where $f(r,\tau)$ *is given by eq.* (3.2), *and within which one may substitute*

(3.14) $$g_m=(\alpha)_m(c_m-1)\,\frac{\Gamma(\beta-\alpha-m-1)}{m!\,\Gamma(\beta-2\alpha-m)},$$

(3.15) *and* $$g(\mu)=\frac{1}{2\pi\,\Gamma(\alpha)}\left|\frac{\Gamma(a_\mu)\,\Gamma(\alpha-a_\mu)}{\Gamma(2i\mu)}\right|^2.$$

Proof: There are two cases to consider when $f_0(r)=1$, namely, (i) $\beta>1$ and (ii) $\beta\le 1$. Suppose (i) $\beta>1$. Since $\phi_0\in\mathcal{L}_{2,\rho}(0,\infty)$, this case is encompassed by the general result of Proposition (3.1). The integrals for g_m and $g(\mu)$ exist and are given in footnote 6. Then, a fair amount of manipulation with Gamma functions yields the formulas (3.14) and (3.15). Next, suppose (ii) $\beta\le 1$. This situation is analogous to the call option payoff for Model I. If we again abbreviate $-\Phi(x,\lambda)=\eta(x,\lambda)I_1(x,\lambda)+\xi(x,\lambda)I_2(x,\lambda)$, then

$$I_1(x,\lambda) = \int_0^x \rho(y)\xi(y,\lambda)dy = \frac{1}{K(\lambda)}\int_0^x y^{(\beta-3)/2-i\sqrt{\lambda}}\exp(-y)M(\alpha-\frac{(\beta-1)}{2}-i\sqrt{\lambda},1-2i\sqrt{\lambda},y)\;dy$$

.

One can show that $I_1(x,\lambda)$ has a pole at $\lambda = \lambda_i = -(\beta-1)^2/4 = -\omega$, with residue $\Gamma(1+\alpha-\beta)/\Gamma(1-\beta)$. Moreover, since $\eta(x,\lambda_i) = U(\alpha,\beta,x)$, the pole generates the additional bound state term $\varphi_p(x,t) = \theta_{(\beta\le 1)}U(\alpha,\beta,x)\Gamma(1+\alpha-\beta)/\Gamma(1-\beta)$ in addition to the already identified contribution $f(r,\tau)$.■

3.2. Economic Interpretation and an Example.

When there is an interest rate equilibrium, one expects a reasonable term structure model solution to behave for large time to maturity as $B(r,\tau) \approx g(r)\exp(-\lambda^*\tau)$, where λ^* is the principal eigenvalue for the (negative of the) operator on the left-hand sides of eqs. (3.1) (or their equivalents in other models). And one expects that λ^* will be strictly positive so that the yield curve, which is given by $R(r,\tau) = -[\ln B(r,\tau)]/\tau$, will *flatten* to a strictly positive rate of interest . Consider eq. (3.13) as $\tau \to \infty$. Think of *A* and *B* as fixed and σ^2 increasing from zero. There are three regimes of behavior depending upon the balance between the parameters: (a) $\sigma^2 < 2AB/(B+2)$; (b) $2AB/(B+2) \le \sigma^2 < 2A$; and (c) $\sigma^2 > 2A$. These regimes can be characterized as follows:

(a) *Equilibrium with the drift term dominant*. In this regime, the deterministic drift term of the underlying stochastic differential equation dominates the (random) volatility term. The equilibrium probability distribution for the interest rates is relatively sharply peaked in the vicinity of *A/B* . From the solution, one sees that there is a point spectrum contribution with the principal eigenvalue given by

$$\lambda^* = \frac{1}{2}\sigma^2(\omega+\lambda_0) = \frac{A}{B} - \frac{\sigma^2}{2B^2}(1+B)$$

(b) *Equilibrium with a balance between the drift and volatility*. In this regime, there is no point spectrum contribution to the yield curve. The strictly positive asymptotic yield is determined by the smallest element of the continuous spectrum, where

$$\lambda^* = \frac{1}{2}\sigma^2\omega = \frac{(2A-\sigma^2)^2}{8\sigma^2}$$

(c) *Non-equilibrium (volatility dominates).* We can see from the previous equation that, as σ^2 approaches 2*A* from below, the asymptotic yield drops to zero. In this regime, one sees from eq. (3.13) that the asymptotic bond price tends to a constant; in this case, there is no stationary rate distribution and the asymptotic yield is always zero. If one considers the growth rate process $Y = \ln r$, then it is easy to see from Ito's lemma that the expected growth rate is always negative in this parameter regime. Hence, although we know the origin is a natural boundary and can never be reached by the risk-neutral *r*-process, there is an almost certain probability of localization of the interest rate r_T near the origin as time grows large.

Two cases, equilibrium and non-equilibrium, are illustrated in Table IV with some typical parameter values. As one sees, with equilibrium, an asymptotic yield exists but the price tends to zero; without equilibrium, an asymptotic price exists, but the yield tends to zero. Hence, we can characterize equilibrium as being associated with a strictly positive principal eigenvalue. These features are expected to be general characteristics of stationary term structure models, including multifactor models.

4. SUMMARY AND EXTENSIONS

We have provided solutions for two relatively old problems by some methods which have general applicability to similar problems in finance. Eigenfunction expansions are very powerful and we have shown how to overcome the main technical difficulty, which is how to generalize the standard method to the case of payoff functions that are not square-integrable with respect to the natural weight function of the problem. Of course, most problems do not admit exact solutions. But, once one realizes the important role played by certain eigenvalues, one is also led to new approximate methods. For example, once one realizes that the yield curve at large times to maturity (in stationary models) is determined by the principal eigenvalue of a differential operator, one can use other techniques (such as variational methods) to explore this regime. These extensions of this work are discussed in Lewis (1994).

Appendix I: *Proof of Proposition 2.1*

The general solution $u(z,\lambda)$ to $\boldsymbol{L}u = \lambda u$ is given by

(A1.1) $$u(z,\lambda) = A\exp(i\sqrt{\lambda}z + \tfrac{1}{2}e^{z})M(a,c,-e^{z}) + B\exp(-i\sqrt{\lambda}z + \tfrac{1}{2}e^{z})M(a-c+1,2-c,-e^{z})\ .$$

In the complex λ-plane, place a branch cut along the real axis to keep $\sqrt{\lambda}$ single-valued. Define the square root so that $\mathrm{Im}\sqrt{\lambda}$ throughout the plane. Along the inversion contour $\mathrm{Im}\sqrt{\lambda} > 0$. In order for $u(z,\lambda)$ to be square-integrable as $z \to -\infty$, we clearly must choose $A = 0$. Hence

(A1.2) $$\xi(x,\lambda) = x^{a} w_2(-x) = \exp[\pi i(1-c) - x]\, x^{1+a-c} M(1-a,2-c,x)\,,$$

where we have used Kummer's transformation[3]. Next, consider $z \to +\infty$ in eq. (A1.1). Using the known asymptotics[4], we have the leading behavior

$$u(z,\lambda) \approx [A\frac{\Gamma(c)}{\Gamma(c-a)} + B\frac{\Gamma(2-c)}{\Gamma(1-a)}]\ \exp[\tfrac{1}{2}e^{z} - \tfrac{1}{2}(\beta-1)z]\ .$$

This last expression cannot be in $\mathcal{L}_2(0,\infty)$ unless we select *A* and *B* such that the expression in the brackets vanishes. From footnote 1, a candidate is

(A1.3) $$\eta(x,\lambda) = x^{a} w_7(-x) = \exp(-x)x^{a}U(c-a,c,x)\ .$$

One can then readily confirm $\eta \in \mathcal{L}_{2,\beta}(X,\infty)$. The Wronskian of the two solutions is

(A1.4) $$W(x) = \frac{\Gamma(2-c)}{\Gamma(1-a)}x^{2a-c}\exp(-\pi ic - x) \equiv K(\lambda)x^{\beta-2}\exp(-x)\ .$$

Evaluate eq.(2.4) by completing the integration contour as indicated. It can be shown that there is no contribution from the large quarter-circles in the limit that they recede to infinity. As shown by Titchmarsh (1962, Chapt. II, Section 2.6) in a more general context, the only singularities of $\Phi(x,\lambda)$ as a function of λ are (i) simple poles on the real λ-axis due to the vanishing of the Wronskian and (ii) the branch point singularity at the origin. (Titchmarsh's demonstration fails for the call option payoff: see Proposition 2.2). The Wronskian vanishes with $K(\lambda)$, which occurs at the poles of $\Gamma(1-a)$; that is, when

(A1.5) $$\lambda_m = -\frac{1}{4}(\beta-3)^2 + m(\beta-3-m)\,, \qquad m = 0,1,2\,,\ldots,[\tfrac{1}{2}(\beta-3)]\,,$$

where $[x]$ denotes the integer part of *x*. This is the point spectrum as stated in the proposition; for it to exist at all, we must have $\beta > 3$. At these points: $a = a_m = 1+m$, $c = c_m = 4-\beta+2m$, and

(A1.6) $$\xi_m(x) \triangleq \xi(x,\lambda_m) = \exp[\pi i(\beta-3) - x]x^{\beta-2-m}M(-m,\beta-2-2m,x)\,,$$

(A1.7) $$\eta_m(x) \triangleq \eta(x,\lambda_m) = \frac{\Gamma(3-\beta+2m)}{\Gamma(3-\beta+m)}\exp(-x)\,x^{\beta-2-m}M(-m,\beta-2-2m,x)\,,$$

so that $\xi_m(x) = k_m \eta_m(x)$. The constant k_m is used below. It is known that confluent hypergeometric functions, when the first argument is a negative integer, are proportional to Generalized Laguerre Polynomials[2]. Specifically, $M(-m,\alpha,x) = [m!/(\alpha)_m]\, L_m^{(\alpha-1)}(x)$, so equivalently,

(A1.8) $$\eta_m(x) = (-1)^m m!\, x^{\beta-2-m} \exp(-x) L_m^{(\beta-3-2m)}(x),$$

which is eq. (2.7) of the proposition. For example, $\eta_0 = \exp(-x)x^{\beta-2}$, $(\beta > 3)$ and $\eta_1 = \exp(-x)x^{\beta-3}[x-(\beta-4)]$, $(\beta > 5)$. Using the Residue Theorem, the point spectrum contribution to $\varphi(x,t)$ is given by

(A1.9) $$\varphi_p(x,t) = \theta_{(\beta>3)} \sum_{m=0}^{[(\beta-3)/2]} \exp\,[-(\delta+\lambda_m)t] \frac{-k_m}{K'(\lambda_m)}\, \eta_m(x) \int_0^\infty y^{-\beta} \exp(y)\, \eta_m(y) \varphi_0(y) dy \ .$$

After some algebra, eq. (A1.9) can be written

(A1.10) $$\varphi_p(x,t) = \theta_{(\beta>3)} \sum_{m=0}^{[(\beta-3)/2]} g_m \eta_m(x) \exp[-(\delta+\lambda_m)t],$$

(A1.11) where $$g_m = \frac{(\beta-3-2m)}{m!\Gamma(\beta-2-m)} \int_0^\infty y^{-\beta} \exp(y)\ \eta_m(y)\ \varphi_0(y)\ dy,$$

which is eq. (2.6) of the proposition. To obtain eq. (A1.11), we have used the formulas

(A1.12) $$K'(\lambda_m) = \exp[\pi i(\beta-m)] \frac{\Gamma(\beta-2-2m)}{(3-\beta+2m)} m! \quad \text{and} \quad (-1)^m = \frac{\Gamma(3-\beta+2m)\Gamma(\beta-2-2m)}{\Gamma(3-\beta+m)\Gamma(\beta-2-m)}.$$

For example, by choosing $\varphi_0 = \eta_k$ in eq. (A1.11) and letting $t = 0$ in eq. (A1.10), one obtains the orthonormality condition

$$\int_0^\infty y^{-\beta} \exp(y)\ \eta_k(y)\eta_m(y)dy = \delta_{km} m! \frac{\Gamma(\beta-2-m)}{\beta-3-2m}.$$

(Note: δ_{km} is the Kronecker delta symbol equal to 1 when the subscripts are equal and zero otherwise).

The continuous part of the expansion is obtained from the integrations above and below the branch cut: since the values of $\Phi(x,\lambda)$ along C_2 and C_3 (see Figure 2) are conjugate, we have

(A1.13) $$\varphi_c(x,t) = \frac{1}{\pi} \int_0^\infty \exp[-(\delta+\lambda)t] \operatorname{Im}[\Phi(x,\lambda)]\, d\lambda \ .$$

It is easy to see that $\eta(x,\lambda)$ is real along the real λ-axis, hence, we need only consider the expressions $\operatorname{Im}[\xi(y,\lambda)/K(\lambda)]$ and $\operatorname{Im}[\xi(x,\lambda)/K(\lambda)]$ when calculating $\operatorname{Im}[\Phi]$ from eq. (2.3). The following manipulations are valid when λ is a positive real number and *x* is real:

(A1.14) $$\mathrm{Im}\Big[\frac{\xi(x,\lambda)}{K(\lambda)}\Big] = -x^{(\beta-1)/2}\mathrm{Im}\Big[\frac{\Gamma(1-a)}{\Gamma(2-c)}x^{-i\sqrt{\lambda}}M(a-c+1,2-c,-x)\Big]$$

$$= \frac{i}{2}x^{(\beta-1)/2}\Big[\frac{\Gamma(1-a)}{\Gamma(2-c)}x^{-i\sqrt{\lambda}}M(a-c+1,2-c,-x) - \frac{\Gamma(c-a)}{\Gamma(c)}x^{i\sqrt{\lambda}}M(a,c,-x)\Big]$$

$$= \frac{-i}{2}\exp(-x)x^{(\beta-1)/2}\,\frac{\Gamma(c-a)\Gamma(1-a)}{\Gamma(c)\Gamma(1-c)}\Big[\frac{\Gamma(1-c)}{\Gamma(1-a)}x^{i\sqrt{\lambda}}M(c-a,c,x) + \frac{\Gamma(c-1)}{\Gamma(c-a)}x^{-i\sqrt{\lambda}}M(1-a,2-c,x)\Big],$$

where we used Kummer's transformation in arriving at the last expression. But the expression in brackets in the last equation is simply $x^{i\sqrt{\lambda}}U(c-a,c,x)$. Hence, along the branch cut:

(A1.15) $$\mathrm{Im}\Big[\frac{\xi(x,\lambda)}{K(\lambda)}\Big] = \frac{-1}{4\sqrt{\lambda}}\Big|\frac{\Gamma[(3-\beta)/2+i\sqrt{\lambda}]}{\Gamma(2i\sqrt{\lambda})}\Big|^2\eta(x,\lambda)\,.$$

Consequently, the continuous spectrum component to $\phi(x,t)$ is given by

(A1.16) $$\varphi_c(\mathrm{x,t}) = \int_0^\infty \exp[-(\delta+\lambda)t]g(\lambda)\eta(x,\lambda)\,d\lambda\,,$$

(A1.17) where $$g(\lambda) = \frac{1}{4\pi\sqrt{\lambda}}\Big|\frac{\Gamma[(3-\beta)/2+i\sqrt{\lambda}]}{\Gamma(2i\sqrt{\lambda})}\Big|^2\int_0^\infty y^{-\beta}\exp(y)\,\eta(y,\lambda)\varphi_0(y)dy\,.$$

Collecting eq. (A1.10) and eq. (A1.16) provides the solution $\varphi(x,t)$ as a linear transformation on any $\varphi_0 \in \mathcal{L}_{2,\rho}(0,\infty)$. Changing integration variables in eq. (A1.16) to $\lambda = \mu^2$ and using $|\Gamma(i\mu)|^2 = \pi/(\mu\sinh\pi\mu)$ then yields Proposition (2.1) as stated. ■

Appendix II: *Proof of Proposition 2.2*

For the call option, $f_0(S) = (S-K)^+$ or $\varphi_0(y) = \gamma(1/y - 1/k)^+$, where recall $\gamma = 2D/\sigma^2$, and $k = \gamma / K$. Hence $\varphi_0 \notin \mathcal{L}_{2,\rho}(0,\infty)$ and so Proposition (2.1) fails for this payoff function because its assumptions do not hold. However, our alternative route is direct computation of $\Phi(x,\lambda)$ from eq. (2.3) for this particular payoff and inspection of its behavior in the λ-plane. As we show below, $\Phi(x,\lambda)$ has additional poles on the real λ-axis beyond those caused by the vanishing of the Wronskian. This is the phenomenom of enlargement of the point spectrum when an operator is extended. For example, see the discussion in Richtmyer (1978, Chapt. 8.4).

Write eq. (2.3) as $-\Phi(x,\lambda) = \eta(x,\lambda) I_1(x,\lambda) + \xi(x,\lambda)\, I_2(x,\lambda)$ and let $\omega(\lambda) \triangleq \gamma\Gamma(1-a)/\Gamma(2-c)$. From known integrals[5] and a parts integration, we find:

(A2.1) $$I_1(x,\lambda) = \int_0^x \rho(y)\xi(y,\lambda)\,\phi_0(y)\,dy = -\omega(\lambda) \int_0^{\min[k,x]} (y^{-a-2} - y^{-a-1} k^{-1})\, M(1-a, 2-c, y)\, dy$$

$$= \begin{cases} \dfrac{-\omega(\lambda)\, k^{-a-1}}{a(a+1)} M(-a-1, 2-c, k) & (k < x) \\ -\omega(\lambda) \left\{ \dfrac{x^{-a-1}}{a(a+1)} M(-a-1, 2-c, x) - \dfrac{(x^{-a-1} - x^{-a} k^{-1})}{a} M(-a, 2-c, x) \right\} & (k > x) \end{cases}$$

$$I_2(x,\lambda) = \int_x^\infty \rho(y)\eta(y,\lambda)\phi_0(y)\,dy = \omega(\lambda)\exp(\pi i c) \int_x^{\max[k,x]} (y^{c-a-3} - y^{c-a-2} k^{-1})\, U(c-a, c, y)\, dy$$

$$= \theta_{(k>x)}\, \omega(\lambda)\exp(\pi i c) \Big\{ \frac{(x^{c-a-2} - x^{c-a-1} k^{-1})}{a(c-a-1)} U(c-a-1, c, x)$$

$$+ \frac{(k^{c-a-2} U(c-a-2, c, k) - x^{c-a-2} U(c-a-2, c, x)}{a(a+1)(c-a-1)(c-a-2)} \Big\}.$$

As one sees, $\Phi(x,\lambda)$ now has two additional simple poles on the negative real λ-axis, at $\lambda = \lambda_i = -(\beta+1)^2/4$, and $\lambda = \lambda_{ii} = -(\beta-1)^2/4$. To maintain the correct inversion integral of eq. (2.4) we must now shift the inversion contour C_1 of Figure 2 to the left of these new singularities. Completing the contour as before, the residues at these poles will contribute new terms to $\phi(x,t)$. That is,

$$\phi(x,t) = \frac{1}{2\pi i} \int_{\chi - i\infty}^{\chi + i\infty} \exp[-(\delta + \lambda)t]\; \Phi(x,\lambda)\, d\lambda = \phi_i + \phi_{ii} + \phi_{iii} + \phi_{iv}$$

where we have denoted the contributions from the two new poles as $\phi_i + \phi_{ii}$, the contribution from the previously identified poles as ϕ_{iii}, and the contribution from the integration along the branch cut as ϕ_{iv}. The remainder of the proof is the computation of these terms.

Residues at λ_i **:** At λ_i, $a=-1$ and $c=-\beta$. Note I_2 contributes nothing because

$$k^{-(\beta+1)}U(-\beta-1,-\beta,k) - x^{-(\beta+1)}U(-\beta-1,-\beta,x) = k^{-(\beta+1)}k^{\beta+1} - x^{-(\beta+1)}x^{\beta+1} = 0.$$

I_1 has residue $-\gamma/\Gamma(1+\beta)$, so $\Phi(x,\lambda)$ has total residue of $-\gamma U(1-\beta,-\beta,x)/\Gamma(1+\beta)x\exp(x)$. This yields

(A2.2) $$\phi_i(x,t) = \gamma\left[\frac{1}{x} - \frac{1}{\beta} + \frac{x^{\beta}}{\beta\Gamma(2+\beta)} M(\beta,\beta+2,-x)\right].$$

In the original variables eq. (A2.2) reads $f_i(S,\tau) = W(S)$, where $W(S)$ is Merton's (1973) term[7].

Residues at λ_{ii} : There are two cases here:

(i) if $\beta>1$, then $i\sqrt{\lambda} = -\frac{1}{2}(\beta-1)$, and so $a=0$ and $c=2-\beta$;

(ii) if $\beta<1$, then $i\sqrt{\lambda} = \frac{1}{2}(\beta-1)$, and so $a=\beta-1$ and $c=\beta$.

First, suppose case (i): I_2 still contributes nothing, and I_1 has the residue $\gamma(1/\beta-1/k)/\Gamma(\beta-1)$. $\Phi(x,\lambda)$ has a total residue $\gamma(1/\beta-1/k)U(2-\beta,2-\beta,x)/\Gamma(\beta-1)\exp(x)$. Second, suppose case (ii): both I_1 and I_2 contribute nothing. We can summarize both cases together with:

(A2.3) $$\phi_{ii}(x,t) = \theta_{(\beta>1)}\,\gamma\left(\frac{1}{\beta}-\frac{1}{k}\right)\left[1-\frac{x^{\beta-1}}{\Gamma(\beta)} M(\beta-1,\beta,-x)\right]\exp(-\beta t).$$

In terms of (S,τ), as $\tau\to\infty$, this yields a term $f_{ii}(S,\tau) \approx g(S)\exp(-r\tau)$. Since all the remaining terms also decay exponentially (see the discussion following Prop. 2.2), this confirms that, indeed, $\lim_{\tau\to\infty} f(S,\tau) = W(S)$.

Residues at the poles of $\omega(\lambda)$ ***and the branch cut contribution*** :

Straightforward but tedious calculations show that

(A2.4) $$\varphi_{iii}(x,t) = \theta_{(\beta>3)}\gamma\, x^{\beta} \sum_{m=0}^{[(\beta-3)/2]} \exp[-x-(\delta+\lambda_m)\,t]\frac{(x\,k)^{-m-2}}{(m+2)!\,\Gamma(d_m)} M(-m-2,c_m,k)M(-m,c_m,x)$$

(A2.5) $$\varphi_{iv}(x,t) = \gamma\,\exp(-x-\delta t)x^{\frac{1}{2}(\beta-1)}k^{-\frac{1}{2}(\beta+1)}$$

$$\times\int_0^{\infty}\frac{1}{4\pi\sqrt{\lambda}}\left|\frac{\Gamma[-\frac{1}{2}(1+\beta)+i\sqrt{\lambda}]}{\Gamma(2i\sqrt{\lambda})}\right|^2 \exp(-\lambda t)(kx)^{i\sqrt{\lambda}}\, U\!\left(\frac{3}{2}-\frac{\beta}{2}+i\sqrt{\lambda},1+2i\sqrt{\lambda},x\right)U\!\left(-\frac{1}{2}-\frac{\beta}{2}+i\sqrt{\lambda},1+2i\sqrt{\lambda},k\right)d\lambda.$$

Substituting the original variables (S,τ), changing integration variable to $\lambda=\mu^2$, and some rearrangement then yields the proposition as stated. ■

Appendix III: *Proof of Proposition 2.5*

We want to establish that for Model I

(A3.1) $$1 = A(S,\tau) + \int_0^\infty G(S,Z,\tau)dZ ,$$

where $A(S,\tau)$ is given by Proposition 2.3 and $G(S,Z,\tau)$ is given by Proposition 2.4. Our method is to (i) take the Laplace transform of eq. (A3.1), and then (ii) show that the resulting equation is equivalent to certain known recurrence relations among the confluent hypergeometric functions $M(a,c,x)$ and $U(a,c,x)$.

First $A(S,\tau) = \widetilde{A}(x,t)$ and $G(S,Z,\tau) = \widetilde{G}(x,y,t)$, where $x = \gamma / S$, $y = \gamma / Z$, and $t = \sigma^2 \tau / 2$. Take the Laplace transform $L[f(t)] = \bar{f}(s)$ of both sides with respect to *t*, where $\mathrm{Re}\, s > 0$. Use $L[1] = 1/s = -1/[\bar{\lambda} + (\beta - 1)^2 / 4]$ in the notation of Proposition 2.3. Also $L[\widetilde{A}(x,t)] = K_{abs}(x,\bar{\lambda})$. Recall that $\exp(-r\tau)G(S,Z,\tau) = \exp(-\beta t)\widetilde{G}(x,y,t)$ is a solution to eq. (2.1) with delta function boundary conditions. The solutions to eq. (2.1) use λ instead of $\bar{\lambda}$, but the two transform variables are related by $\bar{\lambda} = \lambda + \beta$, which is just the interest rate discount factor in terms of the Laplace transform variable. Hence, if we define $\overline{G}$ via $L[\widetilde{G}(x,y,t)] = \overline{G}(x,y,\bar{\lambda})$, then $\overline{G}(x,y,\bar{\lambda}) = \Phi(x,\bar{\lambda})$, where $\Phi(x,\bar{\lambda})$ is given by eq. (2.3) and within which one substitutes $\phi_0(x) = \delta(\gamma / x - \gamma / y)$. In summary, eq. (A3.1) is equivalent to

(A3.2) $$\frac{1}{\bar{\lambda} + \frac{1}{4}(\beta - 1)^2} = -K_{abs}(x,\bar{\lambda}) + \int_0^x \rho(y,\bar{\lambda})\eta(x,\bar{\lambda})\xi(y,\bar{\lambda})dy + \int_x^\infty \rho(y,\bar{\lambda})\xi(x,\bar{\lambda})\eta(y,\bar{\lambda})dy$$

$$= -K_{abs}(x,\bar{\lambda}) + I_1(x,\bar{\lambda}) + I_2(x,\bar{\lambda}).$$

Eqs. (A1.2) and (A1.3) from Appendix I provide $\xi(x,\bar{\lambda})$ and $\eta(x,\bar{\lambda})$ (substitute $\lambda \to \bar{\lambda}$). Using the integration formulas of footnote 5, we find that

(A3.3) $$I_1(x,\bar{\lambda}) = \eta(x,\bar{\lambda}) \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \left[\frac{y^{-\bar{a}}}{\bar{a}} M(-\bar{a}, 2-\bar{c}, y) \Big|_0^x \right]$$

$$= \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \frac{x^{-\bar{a}}}{\bar{a}} M(-\bar{a}, 2-\bar{c}, x)\eta(x,\bar{\lambda}).$$

In eq. (A3.3), the boundary term at $y = 0$ vanishes because $\mathrm{Re}\, s > 0 \Rightarrow \mathrm{Re}\, \bar{a} < 0$. Hence

$$I_1(x,\bar{\lambda}) = \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \frac{\exp(-x)}{\bar{a}} M(-\bar{a}, 2-\bar{c}, x) U(\bar{c}-\bar{a}, \bar{c}, x)$$

$$= \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \frac{1}{\bar{a}} M(\bar{a}-\bar{c}+2, 2-\bar{c}, -x) U(\bar{c}-\bar{a}, \bar{c}, x),$$

where we used Kummer's transformation on the last line[3].

(A3.4) $$I_2(x,\bar{\lambda}) = \xi(x,\bar{\lambda}) \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \exp(\pi i \bar{c}) \left[\frac{y^{\bar{c}-\bar{a}-1} U(\bar{c}-\bar{a}-1, \bar{c}, y)}{(\bar{c}-\bar{a}-1)(-\bar{a})} \Big|_x^\infty \right]$$

$$= \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \exp(\pi i \bar{c}) \Big[\frac{1 - x^{\bar{c}-\bar{a}-1} U(\bar{c}-\bar{a}-1, \bar{c}, x)}{(\bar{c}-\bar{a}-1)(-\bar{a})} \Big] \; \xi(x, \bar{\lambda}) \, .$$

using the asymptotic relation in footnote 4. Hence

$$I_2(x, \bar{\lambda}) = -\frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \frac{1}{\bar{\lambda} + \frac{1}{4}(\beta-1)^2} [x^{\bar{a}+1-\bar{c}} - U(\bar{c}-\bar{a}-1, \bar{c}, x)] M(\bar{a}+1-\bar{c}, 2-\bar{c}, -x)$$

$$= K_{abs}(x, \bar{\lambda}) + \frac{\Gamma(1-\bar{a})}{\Gamma(2-\bar{c})} \frac{1}{\bar{a}(\bar{a}+1-\bar{c})} U(\bar{c}-\bar{a}-1, \bar{c}, x) M(\bar{a}+1-\bar{c}, 2-\bar{c}, -x) \, ,$$

again using Kummer's transformation. Hence eq.(A3.2) is equivalent to

(A3.5) $$\Psi(x) = \Gamma(2-\bar{c})$$

using $$\Psi(x) \triangleq \Gamma(1-\bar{a})[(\bar{a}+1-\bar{c}) M(1,x) U(1,x) + M(2,x) U(2,x)] \, ,$$

$$M(1,x) = M(\bar{a}-\bar{c}+2, 2-\bar{c}, -x) \, , \qquad M(2,x) = M(\bar{a}-\bar{c}+1, 2-\bar{c}, -x) \, ,$$
$$U(1,x) = U(\bar{c}-\bar{a}, \bar{c}, x) \, , \qquad U(2,x) = U(\bar{c}-\bar{a}-1, \bar{c}, x) \, .$$

It is easy to check that $\Psi(0) = \Gamma(2-\bar{c})$, so we need to verify that $\Psi'(x) = 0$, i.e., that

(A3.6) $$(\bar{a}+1-\bar{c})[M(1,x) U'(1,x) - M'(1,x) U(1,x)] + [M(2,x) U'(2,x) - M'(2,x) U(2,x)] = 0 \, .$$

Various recurrence relations may be found in Abramowitz and Stegun (1972, Chapt.13, Sec 13.4).

The relation $$(b-a) M(a-1,b,z) = (b-a-z) M(a,b,z) + z M'(a,b,z)$$

implies $$M(2,x) = -\frac{(x-\bar{a})}{\bar{a}} M(1,x) + \frac{x}{\bar{a}} M'(1,x) \, .$$

The relation $$U(a-1,b,z) = (a-b+z) U(a,b,z) - z U'(a,b,z)$$

implies $$U(2,x) = (x-\bar{a}) U(1,x) - x U'(1,x) \, .$$

The relations $$M'(a,b,z) = \frac{a}{b} M(a+1,b+1,z)$$

and $$(b-a) M(a,b+1,z) = b M(a,b,z) - b M'(a,b,z)$$

imply $$M'(2,x) = -\frac{\bar{a}-\bar{c}+1}{\bar{a}} [M(1,x) - M'(1,x)] .$$

The relations $$U'(a,b,z) = -a U(a+1,b+1,z)$$

and $$U(a,b+1,z) = U(a,b,z) - U'(a,b,z)$$

imply $$U'(2,x) = (\bar{a}-\bar{c}+1)[U(1,x) - U'(1,x)] .$$

Making these substitutions shows that eq. (A3.6) is an identity, which establishes eq. (A3.1). ■

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Footnotes.

1. All the solutions to Kummer's confluent hypergeometric equation $xw_{xx} + (c-x)w_x - aw = 0$, unless c is an integer, are of the form

$$w(x) = A\,M(a,c,x) + B\,x^{1-c}M(a-c+1, 2-c, x) = A\,w_1(x) + B\,w_2(x)\,.$$

Our numbering of solutions follows both Slater (1960) and Erdelyi (1953). $M(a,c,x)$ is the solution defined by the series $\sum_{n=0}^{\infty}(a)_n x^n/(c)_n n!$, where the symbols $(a)_0 = 1$, $(a)_1 = a$, $(a)_2 = a(a+1)$, are employed. Choosing $A = \Gamma(1-c)/\Gamma(a-c+1)$ and $B = \Gamma(c-1)/\Gamma(a)$ yields a solution $U(a,c,x) = w_5(x)$, finite as $x \to \infty$. U is a many-valued function of x, and analytic for all values of a and c. Another solution is $w_7(x) = \exp(x)U(c-a,c,-x)$. In general, the first two parameters a and c for these special functions will be complex numbers. Consider any complex number $z = x + iy = \rho\exp(i\theta)$, where $i = \sqrt{-1}$. We denote the real and imaginary parts, modulus and argument by $x = \operatorname{Re} z$, $y = \operatorname{Im} z$, $\rho = |z|$ and $\theta = \arg z$, respectively.

2. Generalized Laguerre Polynomials are defined by

$$L_m^{(\alpha)}(x) = (1/m!) \quad x^{-\alpha}\exp(x)\left(\frac{d}{dx}\right)^m [x^{m+\alpha}\exp(-x)]\,.$$

3. Kummer's transformation is the identity $M(a,c,x) = e^x M(c-a, c, -x)$.

4. The following asymptotic relations are known:

As $x \to \infty$, $M(a,c,x) = [\Gamma(c)/\Gamma(a)]x^{a-c}\exp(x)\,[(1 + O(|x|^{-1})]$.

As $x \to -\infty$, $M(a,c,x) = [\Gamma(c)/\Gamma(c-a)](-x)^{-a}[(1 + O(|x|^{-1})]$.

As $|x| \to \infty$, $U(a,c,x) = x^{-a}[(1 + O(|x|^{-1})]$, $\quad (-3\pi/2 < \arg x < 3\pi/2)$.

5. The following indefinite integrals are known (Slater, 1960, Sec 3.2):

$$\int x^{a-2}M(a,c,x)dx = \frac{x^{a-1}}{a-1}M(a-1,c,x) + C \quad (a \neq 1)\,.$$

$$\int x^{a-2}U(a,c,x)dx = \frac{x^{a-1}}{(a-1)(a-c)}U(a-1,c,x) + C \quad (a \neq 1 \text{ and } a \neq c)$$

6. The following definite integral is known (Slater, 1960, Sec 3.2.3):

$$\int_0^{\infty} x^{b-1}\exp(-x)\,U(a,c,x)dx = \frac{\Gamma(b)\Gamma(1+b-c)}{\Gamma(1+a+b-c)}\,. \quad (\operatorname{Re}(b) > 0, \quad \operatorname{Re}(c) < \operatorname{Re}(b) + 1)$$

7. Merton's asymptotic solution to eq. (2.1) for the call option is (Merton, 1973, eq. 8.46):

$$W(S) = S - \frac{D}{r} + \left(\frac{2D}{\sigma^2}\right)^{\beta+1}\frac{S^{-\beta}}{\beta\Gamma(\beta+2)}M\left(\beta, \beta+2, \frac{-2D}{\sigma^2 S}\right), \text{ where } \beta = 2r/\sigma^2\,.$$

8. The formula may be derived by making the substitution, where the signs of $\pm i\infty$ and c are opposite ,

$$\frac{1}{\nu + ic} = \int_0^{\pm i\infty} \exp[-s(\nu + ic)]\, ds \,.$$

9. In writing down the expression for $A_\infty(S)$, we used $\frac{x^\beta}{\Gamma(\beta+1)} M(\beta, \beta+1, -x) = 1 - \frac{\Gamma(\beta, x)}{\Gamma(\beta)}$.

Table I

Comparisons between the Call Option Values
for the Constant Yield and Constant Dividend Models.

Time to Maturity (years)	Stock Price: S = $90		S = 100		S = 110	
	Constant Yield	Constant Dividend	Constant Yield	Constant Dividend	Constant Yield	Constant Dividend
.01	0.00	0.00	0.80	0.80	10.00	10.00
.1	0.12	0.12	2.51	2.52	10.19	10.19
.2	0.48	0.49	3.53	3.55	10.67	10.68
.3	0.89	0.91	4.30	4.34	11.17	11.20
.4	1.29	1.33	4.94	4.99	11.65	11.69
.5	1.67	1.72	5.50	5.57	12.10	12.16
1.0	3.26	3.42	7.58	7.77	13.94	14.12
2.0	5.44	5.92	10.18	10.71	16.43	16.95
3.0	6.89	7.75	11.84	12.78	18.08	19.03
4.0	7.90	9.19	12.98	14.39	19.22	20.67
5.0	8.62	10.37	13.78	15.70	20.02	22.01
10.0	9.86	14.03	15.05	19.75	21.16	26.24
20.0	8.20	16.79	12.70	22.85	18.05	29.56
100.0	0.20	18.13	0.46	24.41	0.89	31.29
Infinite	0.00	18.14	0.00	24.41	0.00	31.29

Notes: For this table, the interest rate r = 5% per year, the volatility σ = 20% per $\sqrt{\text{year}}$, the dividend D = \$5 per year, and the exercise price K = \$100. The constant yield parameter y is always \$5/S.

Table II

Differences between the Call Option Values
for the Constant Yield and Constant Dividend Models .

Annual Dividend *D*	Strike Price: K = \$50		K = 100		K = 150	
	Volatility: σ = .2	σ = .4	σ = .2	σ = .4	σ = .2	σ = .4
\$1.00	.020	.032	.050	.088	.009	.058
5.00	.001	.082	.193	.385	.028	.250
10.00	-.235	-.015	.275	.648	.031	.405

Notes: For this table, the interest rate r = 5% per year, the time to maturity τ = 1 year, and the stock price S = \$100. The constant yield parameter is always $y = D/\$100$, where D is the annual dollar dividend.

Table III

Put Option Values for the Constant Dividend Model.

Time to Maturity (years)	Stock Price: S = $90			S = 100			S = 110		
	Part 1	Part 2	Total Put Value	Part 1	Part 2	Total Put Value	Part 1	Part 2	Total Put Value
1.0	0.00	13.42	13.42	0.00	7.77	7.77	0.00	4.12	4.12
2.0	0.00	15.92	15.92	0.00	10.71	10.71	0.00	6.95	6.95
3.0	0.00	17.75	17.75	0.00	12.78	12.78	0.00	9.03	9.03
4.0	0.00	19.19	19.19	0.00	14.39	14.39	0.00	10.67	10.67
5.0	0.00	20.37	20.37	0.00	15.70	15.70	0.00	12.01	12.01
10.0	1.76	22.17	23.93	0.94	18.76	19.70	0.51	15.70	16.21
20.0	14.22	8.04	22.26	11.36	8.20	19.56	9.06	8.10	17.16
100.0	0.57	0.003	0.57	0.54	0.004	0.54	0.51	0.005	0.52
Infinite	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Notes: The total put value is broken into 2 pieces: Part 1 is the bankruptcy claim due to the stock price being absorbed at zero. Part 2 is the put value conditional on no absorption. For this table, the interest rate r = 5% per year, the volatility σ = 20% per $\sqrt{\text{year}}$, the dividend D = $5 per year, and the exercise price K = $100.

Table IV

Bond Prices and Yields for Model II:
Equilibrium and Non-Equilibrium Cases.

Years to Maturity	Equilibrium Example: Short Term Rate r = 3% per year, Volatility σ=.25 per $\sqrt{\text{year}}$.		Non-Equilibrium Example: r = 3% per year, σ=.75 per $\sqrt{\text{year}}$.	
	Price	Yield (%)	Price	Yield (%)
1	0.9692	3.133	0.9694	3.106
2	0.9369	3.262	0.9391	3.144
4	0.8694	3.498	0.8850	3.056
5	0.8351	3.605	0.8621	2.967
10	0.6688	4.023	0.7818	2.462
20	0.4090	4.470	0.7058	1.742
30	0.2456	4.680	0.6714	1.328
100	$6.658\ 10^{-3}$	5.012	0.6213	0.476
Infinite	0.0	5.156	0.6156	0.0

Notes: For this table, the parameter $A = .15$ per year and $B = 2.0$ per year.
(Figures are attached as hard copy only)