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"A Simple Option Formula for General Jump-Diffusion and other Exponential Lévy Processes"

Alan L. Lewis

Overheads and additional notes to be posted at www.optioncity.net (Publications)

Topics

Why jump-diffusion models?

European-style options.

1. **Solutions in "Stock price" space are complicated**
2. **The "Fourier-space" solution is simple**
3. **Moving integration contours around is useful.**

American-style options

4. **Simple numerical method (method of lines)**
5. **The analytic $T \to \infty$ solutions**

Why Jump-Diffusion Models for Options?

I. Benchmark model (exponential Brownian motion):

Attractive features:

- **limited liability stock prices**
- **uncorrelated, level independent returns**
- **simple formulas (methods) for option prices (euro,amer)**

Weak points:

- **Actual stock price distributions have wider tails**
- **Lacks volatility clustering (auto-corr. of absolute returns)**
- **Lacks stock price jumps**
- **Poor fit to real-world option prices (smile/skew)**

II. Jump-diffusion generalization (exponential Lévy processes)

- Class of all stationary, independent increment processes
- Subclass: Brownian motion plus Poisson jumps

Attractive features:

- **all the attractive benchmark features +**
- **large flexible class of models, each with a few parameters**
- **wide return tails common (exponential decay, moments)**
- **Good fits to expiring options (fear of jumps/crashes?)**

Weak points:

- **Lacks volatility clustering (auto-corr. of absolute returns)**
- **Brownian motion $\approx$ Large number of small jumps**

Stock price Evolution and Examples

$$S_t = S_0 \exp(X_t),$$

(Assumption: this is under the martingale pricing measure Q)

where $X_t = ct + \sigma B_t + \Delta X_t$;

$\Delta X_\tau = y \sim \mu(y)$ **(a measure)**

Type A: Poisson sub-class $\int_{-\infty}^{\infty} \mu(y)dy < \infty$

$\mu(y) = \lambda p(y)$, **where** $\int_{-\infty}^{\infty} p(y)dy = 1$

Examples:

(A.1) Merton's 1976 jump-diffusion model with log-normally distributed jumps:

$$p(y) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-(y-\alpha)^2 / 2\delta^2\right]$$

(A.2) Kou's 2000 jump-diffusion model with exponentially distributed jumps:

$$p(y) = \frac{1}{2\eta} \exp\left[-\,|\,y-\kappa\,|\,/\,\eta\right]$$

Stock price Evolution and Examples (cont.)

$$S_t = S_0 \exp(X_t),$$

where $X_t = ct + \sigma B_t + \Delta X_t$;
$\Delta X_\tau = y \sim \mu(y)$ (a measure)

Type B: No Poisson intensity exists: $\int_{-\infty}^{\infty} \mu(y)dy = \infty$

Example:

(B.1) Carr and Wu's (2000) "Finite Moment Logstable Process"

$$\mu(y) = \frac{c_\pm}{|y|^{1+\alpha}}, \quad c\pm = \begin{cases} c_+, & y > 0 \\ c_-, & y < 0 \end{cases} \quad 1 < \alpha < 2$$

($\alpha = 2$ is Brownian motion)

European-style options.

Solutions in "Stock price" space are complicated

Example: Madan, Carr, and Chang's "Variance Gamma Process" Pure jumps: BM sampled at random times

The Call Option Price:

$$C(S_0) = S_0 \Psi\left(d\sqrt{\frac{1-c_1}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_1}}, \frac{\tau}{\nu} \right)$$
$$-Ke^{-r\tau}\Psi\left(d\sqrt{\frac{1-c_2}{\nu}}, (\alpha+s)\sqrt{\frac{\nu}{1-c_2}}, \frac{\tau}{\nu} \right),$$

where $d = \frac{1}{s}\left[\log\left(\frac{S_0}{K}\right) + r\tau + \frac{\tau}{\nu}\log\left(\frac{1-c_1}{1-c_2}\right) \right]$

and

$$\Psi(a,b,\gamma) = \frac{c^{\gamma+1/2}\exp(sign(a)c)(1+u)^{\gamma}}{\sqrt{2\pi}\Gamma(\gamma)} X$$
$$+K_{\gamma+1/2}(c)\Phi\left(\gamma, 1-\gamma, 1+\gamma; \frac{(1+u)}{2}, -sign(a)c(1+u)\right)$$
$$+ \ldots \text{ (4 more lines)}$$

European-style options (cont.)
Solutions in "Fourier" space are simple

Ingredients:

1. The generalized Fourier transform of the payoff function:
For the call option:

$$\hat{w}(z) = \int_{-\infty}^{\infty} e^{izx}(e^x - K)^+ dx = -\frac{K^{iz+1}}{(z^2 - iz)}, \quad \operatorname{Im} z > 1$$

2. The characteristic function of the Lévy process:

$$\varphi_T(z) = \int_{-\infty}^{\infty} e^{izx} p_T(x) dx = \mathbb{E}[\exp(izX_T)] = \exp\big(-T\Psi(z)\big)$$

where $\Psi(z)$ is the "characteristic exponent".
For the VG model example:

$$\varphi_T(z) = \exp(ic\omega T)\left(1 - iz\nu\theta + \frac{1}{2}\sigma^2\nu z^2\right)^{-T/\nu}, \quad \alpha < \operatorname{Im} z < \beta$$

Then, the option price is given by $(y = \log(S_0))$

$$V(S_0) = \frac{e^{-rT}}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} e^{izy} \varphi_T(-z) \widehat{w}(z) dz, \quad \nu \text{ has } \underline{\text{conditions}}$$

The integration is along a line parallel to the real z-axis.
(Closely related results: Carr & Madan, Bakshi & Madan, Raible)

Generalized Fourier Transforms for various Payoffs

Financial Claim (Option)	Payoff Function: $w(x)$	Payoff Transform: $\hat{w}(z)$	Strip of regularity $\mathcal{S}_w$
Call option	$\left(e^x - K\right)^+$	$-\dfrac{K^{iz+1}}{z^2 - iz}$	$\operatorname{Im} z > 1$
Put option	$\left(K - e^x\right)^+$	$-\dfrac{K^{iz+1}}{z^2 - iz}$	$\operatorname{Im} z < 0$
Covered call/ cash-secured put	$\min\left(e^x, K\right)$	$\dfrac{K^{iz+1}}{z^2 - iz}$	$0 < \operatorname{Im} z < 1$
Delta function	$\delta(x - \ln K)$	K^{iz}	Entire z-plane
Money market	1	$2\pi\delta(z)$	$\operatorname{Im} z = 0$

European-style options (cont.)
Solutions in "Fourier" space are simple

The solution is very easy to derive and "obvious":

First, we need the inversion formula for the payoff function:

$$w(x) = \frac{1}{2\pi}\int_{i\nu-\infty}^{i\nu+\infty} e^{-izx}\hat{w}(z)dz, \quad x = \log S_T, \quad z \in \text{Payoff strip}$$

Then, by martingale pricing:

$$V(S_0) = e^{-r\tau}\mathbb{E}[\,w(\log S_T)\,] = \frac{e^{-r\tau}}{2\pi}\mathbb{E}\left[\int_{i\nu-\infty}^{i\nu+\infty} e^{-iz\log S_T}\,\widehat{w}(z)dz\right]$$

$$= \frac{e^{-r\tau}}{2\pi}\mathbb{E}\left[\int_{i\nu-\infty}^{i\nu+\infty} e^{-iz\log S_0}e^{-izX_T}\,\widehat{w}(z)dz\right]$$

$$= \frac{e^{-rT}}{2\pi}\int_{i\nu-\infty}^{i\nu+\infty} e^{iz\log S_0}\varphi_T(-z)\widehat{w}(z)dz\ ,$$

Ok to exchange the integrations (sufficient conditions) if:

1. $w(x)$ **is Fourier integrable in some Payoff strip** S_w **and bounded for** $|x| < \infty$.
2. $\varphi_T(-z)$ **is regular in some strip** S_X^*: $\alpha < \operatorname{Im} z < \beta$
3. $\nu = \operatorname{Im} z$ **lies in the intersection of these two strips**

European-style options (cont.)
Moving integration contours is easy

<u>In practice</u>: for typical financial claims:

There is complete freedom to integrate anywhere in the strip of regularity of the characteristic function (Residue Theorem)

Example: the call option with strike *K*.

Let $\varphi_T(-z)$ be regular in some strip $S_X^*: \alpha < \operatorname{Im} z < \beta$, where $\alpha < 0$ and $\beta > 1$. Then,

$$C(S_0) = \frac{Ke^{-rT}}{2\pi} \int_{i\nu_1 - \infty}^{i\nu_1 + \infty} e^{izk} \varphi_T(-z) \widehat{w}(z) \frac{dz}{z^2 - iz},$$

where $1 < \nu_1 < \beta$, and $k = \log(S_0 / K)$

Now move contour to $0 < \nu_2 < 1$. There are simple poles at $z = 0$ and $z = i$. You will pick up a residue at $z = i$.

$$C(S_0) = Se^{-qT} - \frac{Ke^{-rT}}{2\pi} \int_{i\nu_2 - \infty}^{i\nu_2 + \infty} e^{izk} \varphi_T(-z) \widehat{w}(z) \frac{dz}{z^2 - iz},$$

where $0 < \nu_2 < 1$.

Applications: Black-Scholes style formulas,
Asymptotics (*T*, *K*, other parameters $\to \infty$).

II. American-style options (Put option example)

The problem: determine the optimal exercise strategy (a stopping time strategy)

$$V(S_0,T) = \max_{0 \le \tau(\omega) \le T} \mathbb{E}\Big[e^{-r\tau}(K - S_\tau)^+\Big],$$

For simplicity: $S_t = S_0 \exp(X_t)$, **where** X_t **is a jump-diffusion (a Poisson intensity** λ **exists). By homogeneity, the solution is**

$$V(S_0,T) = K\, f(x,T), \quad \text{where } x = \log(S_0 / K) \text{ and}$$

The reduced problem: find $b(t) \le 0$ **and** $f(x,t)$**, where**

(i) for $b(t) \le x < \infty$, using $k = \int_R (e^y - 1)\, p(y)\, dy$

$$f_t = \tfrac{1}{2}\sigma^2 f_{xx} + (r - \tfrac{1}{2}\sigma^2 - \lambda k) f_x - (r+\lambda) f + \lambda \int_R f(x+y) p(y) dy$$

(ii) for $-\infty < x \le b(t), \quad f(x,t) = 1 - e^x$

Subject to:

(i) $f(x,0) = (1 - e^x)^+$

(ii) $f(x = b(t), t) = 1 - e^{b(t)}$

(iii) $f_x(x = b(t), t) = -e^{b(t)}$ (smooth pasting).

(iv) $f(x,t) \to 0$ as $x \to \infty$

A good numerical method for this problem (G.H. Meyer): The Method of Lines. Write $f_t = (f_n - f_{n-1}) / \Delta t$**. Take** $\Delta t = T / N$**, and just keep solving ODEs for** $n = 1, 2, \ldots, N$**. (Actually there is a sub-iteration at each** n **for the jump term).**

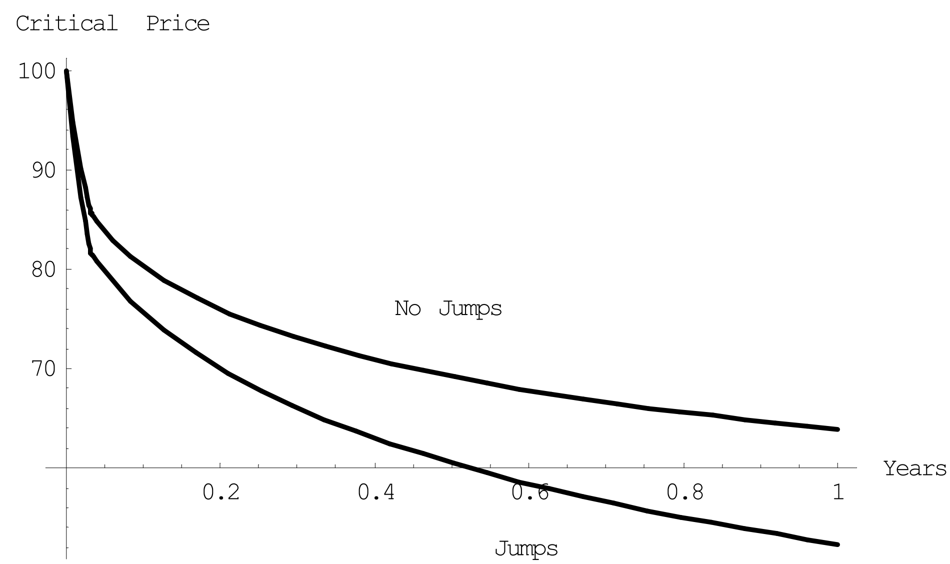
II. American-style options (Put option example)

Numerical example: K=100 $r = 0.08$ $\sigma = 0.40$ $T = 1$ year.
Jumps: frequency $\lambda = 1$; Two possible jumps:

$$e^y = \begin{cases} \textbf{1.25 (+25\%) prob=1/2} \\ \textbf{-0.50 (-50\%) prob=1/2} \end{cases}$$

Results from Method of Lines computation:

Critical Price



Time to Expiration

II. American-style options (Put options)

Numerical example continued:
Let $T \to \infty$ in the Method of Lines program (you can!)

Numerical Results for the critical boundary

No Jumps: $S^* = 50$ With Jumps: $S^* \cong 32.16$
Both results confirmed by exact analytic formulas:

The $T \to \infty$ Perpetual Put Is Completely Solved Analytically

Case	Difficulty	By whom:
Brownian motion +		
I. Up jumps	Easy	(? Wald)
II. Down jumps	Moderate	Gerber, Landry, & Shiu
III. Up & Down jumps	Hard	Boyarchenko & Levendorskii

Formula, case III: (easy integration, once you know answer!) :

$$S^* = K\varphi^{-}(-i) = K\exp\left[-\frac{1}{2\pi}\int_{i\omega-\infty}^{i\omega+\infty}\frac{\log\left[r+\Psi(z)\right]}{z(z+i)}dz\right] = 32.16$$

where $0 < \omega <$ Zero of $\left[r + \Psi(iy)\right]$, $y =$ real.
Will post on web site: Very direct derivation of case II.

American-style Put options: perpetual case boundary

Moderately hard case: Brownian motion + negative jumps.
3 steps: much faster than Gerber, Landry & Shiu,
(following suggestion by David Dickson, a discussant)

<u>Step 1</u>. Translate $x' = x - b$ (now relabel $x' \to x$) and introduce

$$G(x) = \begin{cases} f(x) - (1 - e^{x+b}) & x \ge 0 \\ 0 & x \le 0 \end{cases},$$

which satisfies, on $x \ge 0$, with $c = r - \frac{1}{2}\sigma^2 - \lambda k$, $h(y) = p(-y)$,

$$(*) \qquad r = \tfrac{1}{2}\sigma^2 G'' + cG' - (r+\lambda)G + \lambda \int_0^x G(x-y)h(y)dy$$

B.C.: (i) $G'(0) = G(0) = 0$,

(ii) $G(x) \approx e^{x+b} - 1 +$ vanishing terms, as $x \to \infty$

<u>Step 2</u>. Solve (*) with Laplace transform $\hat{G}(s) = \int_0^\infty e^{-sx} G(x)dx$

$$\frac{r}{s} = \psi(s)\hat{G}(s), \quad \text{where the } \textit{Laplace} \text{ exponent}$$

$$\psi(s) = \tfrac{1}{2}\sigma^2 s^2 + c\,s - (r+\lambda) + \lambda \hat{h}(s)$$

invert

$$\Rightarrow \qquad G(x) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\chi + i\infty} \frac{r}{s\psi(s)} e^{sx} ds \quad , \quad \chi > 1 \text{ (right-most pole)}$$

<u>Step. 3</u>. Move the contour to $\chi' < 0$. The martingale condition causes $\psi(s=1) = 0$. The residue at $s = 1$ must be e^b to satisfy B.C. (ii). This determines $e^{b^*} = r/\psi'(s=1)$ or $S^* = Kr/\psi'(1)$.

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