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"Introduction to Mathematical Finance for Science Students"

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Overheads to be posted at www.optioncity.net (Publications)

Topics

1. The Nobel prizes for mathematical finance.

2. Bachelier – the "father" of math. finance

3. More on the Black-Scholes option theory.
   (i) Itô's stochastic calculus.
   (ii) Probabilistic solutions of PDEs.
   (iii) Derivation of the Black-Scholes formula.
   (iv) Generalizations: fundamental theorems.

4. Problems with the formula: option "smiles".
   (i) possible fixes: stochastic volatility
   (ii) even more fixes: diffusions + jumps

The Nobel prizes.

Bank of Sweden Prize in Economics in Memory of Alfred Nobel.

The “finance” prizes:

(i) 1990: Harry Markowitz, William Sharpe (portfolio theory), and Merton Miller (fin. Econ.)

(ii) 1997: Myron Scholes and Robert Merton – Black-Scholes option theory. (Fischer Black 1938-1995).

(iii) 2003: Clive Granger (time series methods), and Robert Engle (volatility modeling/GARCH).

Louis Bachelier –Father of Mathematical Finance

- **Ph.D student of Poincaré (Paris/Sorbonne)**
- **His 1900 thesis was titled "Theory of Speculation".**
- **(pre-dates Einstein's 1905 Brownian motion paper.)**

Thesis summary:

- **A Brownian motion model for stock prices (Paris exchange).**
- **A mathematical physics thesis in an unrecognized subject.**
- **The purpose of the model was to value options.**

Recall the characteristics of a Brownian motion: $X_t \equiv \sigma B_t$?

- **Ex. of: a stochastic process, a diffusion, a martingale.**
- X_t **= location of the Brownian "particle" at time** t.
- X_t **is continuous (but not differentiable), with transition density:**

  $$p(t,x,y)dy = P_x(X_t \in dy) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\{(x-y)^2/(2\sigma^2 t)\}}$$

  Satisfies the heat equation $\dfrac{\partial p}{\partial t} = \dfrac{\sigma^2}{2}\dfrac{\partial^2 p}{\partial x^2}$

  Initial condition: $p(0,x,y) = \delta(x-y)$ **(Dirac delta).**

- **Martingale Property:** $E_x[X_t] = \int_{-\infty}^{\infty} y\, p(t,x,y)dy = x = X_0$

  [Martingales: $E_x[X_t] = X_0$, **driftless** $\Rightarrow E_x[dX_t] = 0$**]**

- $E_x[(X_t - x)^2] = \sigma^2 t$ **(**σ^2 **is the volatility/diffusion coef.)**

  [Itô algebra: $E_x[(dX_t)^2] = \sigma^2 dt$**]**

  [Stochastic Differential Eqn (SDE): $dX_t = (0)dt + \sigma dB_t$**]**

Bachelier's Option Pricing Model

<u>What is an option</u>? (call options, put options, other types).

- **A "call" option is a security (contract). It gives the owner, the right, but not the obligation, to purchase an underlying security at a fixed price (the strike K) on an expiration time T.**
- **S_t = Underlying security price at time t. (observable)**
- $C_T = \begin{cases} 0, & \text{if } S_T \le K \\ S_T - K, & \text{if } S_T \ge K \end{cases} = \max(0, S_T - K) = (S_T - K)^+$
- **The option may be "exercised" only at expiration (Euro-style), but trades in the market place at every time $t < T$.**
- **The <u>Problem</u>: C_t = Call option price at time $t < T$.**
  (Is there a formula?). (see graph in last slide)

<u>Bachelier's Solution:</u>

The Option value = expected value = fair game value:

$$\text{With: } p(T, S_0, S_T) = \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\{(S_0 - S_T)^2/(2\sigma^2 T)\}} \Rightarrow$$

$$C_0 = C(S_0, T, \sigma^2) = E[\,C_T \mid S_0\,] = \int (S_T - K)^+ p(T, S_0, S_T) dS_T$$

Excerpt from:
"Bachelier and his Times:
A Conversation with Bernard Bru " (Univ. Paris V)
by Murad S. Taqqu (Boston University). (April 25, 2001).

Taqqu.: The subject of Bachelier's thesis was out of the ordinary.

Bru: In fact, it was exceptional. On the other hand, Bachelier was the right man at the right time, first because of his experience in the stock exchange. Secondly, he knew the theory of heat (this was the height of classical mathematical physics). Third, he was introduced to probability by Poincaré and he also had the probability lecture notes of Joseph Bertrand, which served him well. If you look at Bertrand's chapter on gambling losses, you will see that it was useful to Bachelier. But the idea of following trajectories is attributable to Bachelier alone. It's what he observed at the stock exchange.

Kiyoshi Itô's Stochastic Calculus (1944, Kyoto Univ.)

Associated with each (homogeneous) diffusion process is an evolution equation (SDE):

$$\{X_t\} \Leftrightarrow dX_t = b(X_t)dt + a(X_t)dB_t, \quad (1)$$

or in integrated form:

$$X_T = x + \int_0^T b(X_t)dt + \int_0^T a(X_t)dB_t,$$

where $b(X_t)$ **= instantaneous drift rate**

$a^2(X_t)$ **= instantaneous volatility (diffusion coef.)**

B_t **= standard (mean=0, vol.=1) Brownian motion process**

<u>Ito's lemma</u>: Suppose $f(x,t)$ is some smooth (twice-dif.) function and $\{X_t\}$ is a diffusion with the SDE above.

Then, $\{f_t\} \equiv f(X_t,t)$ is also a diffusion and has the SDE:

$$df_t = \left\{\frac{\partial f}{\partial t} + b(X_t)\frac{\partial f}{\partial x} + \frac{1}{2}a^2(X_t)\frac{\partial^2 f}{\partial x^2}\right\}dt + \left\{a(X_t)\frac{\partial f}{\partial x}\right\}dB_t.$$

Proof: Use Taylor expansion, eqn. (1), and the rule $(dB_t)^2 = dt$.

Neglect all terms of $O(t^{3/2})$ or higher. (see reading list).

Two Immediate Applications of Itô's lemma

1. (elliptic) PDE solutions can be promoted to martingales.

Remember we said that any diffusion of the form $dm_t = (0)dt + a(X_t)dB_t$ is a martingale.
So, Ito's result says: take any solution $f(x,t)$ of the PDE

$$0 = \frac{\partial f}{\partial t} + b(X_t)\frac{\partial f}{\partial x} + \frac{1}{2}a^2(X_t)\frac{\partial^2 f}{\partial x^2}$$

Then, $f_t = f(X_t, t)$ must be a martingale: $f_0 = E[f_T]$.
(More generally, m_t is a martingale if $m_t = E[m_T \mid \mathscr{F}_t]$, $T \geq t$), where $\mathscr{F}_t$, the "filtration", describes how infomation is revealed).

2. Conversely, PDEs can be "solved" by running a diffusion.
(1920's onward: 1940's: Itô, Kakutani, Feynman, Kac)

Suppose we want to solve the PDE problem: $x \in \mathbb{R}$, $t < T$:

$$0 = \frac{\partial f}{\partial t} + b(x)\frac{\partial f}{\partial x} + \frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2},$$

$f(x,T) = \varphi(x)$, a given function.

$f(x,0) = ?$, the unknown function

The "probabilistic solution": First, run the diffusion:

$$dX_t = b(X_t)dt + a(X_t)dB_t, \qquad X_0 = x.$$

Then, the PDE solution is:

$$f(x,0) = E_x[\varphi(X_T)].$$

Criticisms of the Bachelier formula

1. His stock price probability density (normal density) $p(T, S_0, S_T)$ allows $S_T < 0$, but stocks are limited liability securities.

Response: the simplest fix, change $dS_t = \sigma dB_t$ to

$$dS_t = \alpha S_t\,dt + \sigma S_t\,dB_t \quad (\alpha, \sigma \text{ are constants})$$

(Geometric Brownian motion, M.F.M. Osborne, 1959). Then, $p(T, S_0, S_T)$ is a "log-normal" density.

Further challenges: actual price distributions are similar to the log-normal, but with much wider 'tails', kurtosis $>> 3$ (Eugene Fama dissertation, 1965).

2. The "fair game" (mathematical expectation) approach to security prices is flawed. People are "risk-averse" and will not pay the expected value for risky gambles.

Classic ex: St. Petersburg game paradox (D. Bernoulli, 1738). A fair coin is tossed N times until heads appears. The Payoff to you is 2^N dollars. What would you pay to play? (The expected payoff is $\frac{1}{2}2 + \frac{1}{4}2^2 + \frac{1}{8}2^3 + ... = \infty$).

Bernoulli's solution: risky payoffs W are valued with a 'utility' $u(W) = \log W < W \Rightarrow$ finite value for the game.

More general: each rationale person (should, does) max $E[U] = \sum p_i U(W_i)$=expected utility. ($U' > 0, U'' < 0$)

The Black-Scholes (1973) solution to the Option Pricing Problem

Assumptions.

1. Frictionless, arbitrage-free, continuous-time markets.
2. There is a riskless money-market security M_t paying rate r.
   If you buy it, your wealth grows $M_t = M_0 e^{rt}$, ($dM_t = rM_t dt$).
3. There is stock following GBM: $dS_t = \alpha S_t\, dt + \sigma S_t\, dB_t$.
4. There is a call option with price $C(S_t, t)$=? (unknown), which trades in the market place for all $t < T$ = expiration.

Solution

You can create a synthetic money-market security M_t by buying f shares of stock and sell 1 call option.

Your wealth $M_t = f\, S_t - C_t$ changes in the next instant dt, by

$dM_t = f\, dS_t - dC_t$ = (by Ito's lemma)

$$f\,\{\alpha S\, dt + \sigma S\, dB_t\}$$
$$-\left\{\frac{\partial C}{\partial t} + \alpha S \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right\} dt - \sigma S \frac{\partial C}{\partial S} dB_t$$

The position is made riskless by choosing $f = \partial C / \partial S$.

Since its riskless, we must have $dM_t = rM_t dt = r(f\, S - C)dt \Rightarrow$

$$r\left\{S\frac{\partial C}{\partial S} - C\right\} dt = -\left\{\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right\} dt$$

The Black-Scholes (1973) solution (cont.)

Rearranging $\Rightarrow$
<u>Black-Scholes PDE</u>:

$$0 = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC$$

with the terminal condition $C(S_T,T) = \varphi(S_T) = (S_T - K)^+$.
The PDE is easily solved by our previous general formula:

$$C(S_0,0) = e^{-rT} E[\varphi(S_T) \mid S_0] = e^{-rT}\int (S_T - K)^+ q(T,S_0,S_T)dS_T$$

<u>Now</u> $q(T,S_0,S_T)$ is the transition density for the SDE:

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t. \quad \text{(Notice: } \alpha \to r\,!)$$

$q(T,S_0,S_T)$ is the density of the <u>risk-neutral measure Q,</u> or <u>equivalent martingale measure Q.</u> Why? Because $d(e^{-rt}C_t) =$ pure noise/Q $\Rightarrow$Q-martingale. The real-world measure P, had density $p(T,S_0,S_T)$. 'Equivalent', $(P \sim Q)$, means they have the same set of possible $(S_T > 0)$ and impossible $(S_T \leq 0)$ events.

<u>Black-Scholes formula</u>:

$$C(S_0,0) = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where $\Phi(z) = \int_{-\infty}^{z} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ = cumulative normal

and $$d_\pm = \frac{\log(S_0 / Ke^{-rT}) \pm \sigma^2 T/2}{\sigma\sqrt{T}}$$

Final Comments on the Black-Scholes Solution

We showed, similar to B-S, that we could achieve:
Money market account = long stock position – call option.
The more modern viewpoint, although equivalent:
Call option = long stock position – borrowing;
i.e., the call option may be "dynamically replicated" in a world that fits with the B-S assumptions.

Defn. A <u>complete market</u> is one in which any derivative security may be dynamically replicated by similar trading in the underlying. If a market is not complete, it is called <u>incomplete</u>. Real option markets are generally acknowledged to be incomplete. Some sources of incompleteness: stochastic volatility, and/or price jumps.

We showed that $d(e^{-rt}C_t) =$ pure noise. That is, the discounted call option price is a Q-martingale. The modern viewpoint, is that there is always some arbitrary 'numeraire' security , such as the money market account (bond) $M_t = M_0 e^{rt} > 0$ and $C_t / M_t =$Q-martingale.

Harrison and Pliska (1981). (Fundamental Theorems of Asset Pricing). Baxter and Rennie paraphrase:
Suppose a market of securities and a numeraire bond. Then,
(1) The market is arbitrage-free if and only if there is at least one equivalent martingale measure Q.
(2) The market is complete if and only if Q is unique.

Some reading

On Bachelier:

Murad S. Taqqu , *Bachelier and his Times, A Conversation with Bernard Bru,* 2001, (math.bu.edu/people/murad/pub/ bachelier-english43-fin-posted.pdf)

Black-Scholes-Merton Theory

F. Black and M. Scholes, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy, 1973, 81, May 637-59.

R.C. Merton, *Continuous-time Finance*, 1990, Basil Blackwell.

Arbitrage-free Theory

J.M. Harrison and S.R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Processes and their Applications, 11, 1981, 215-260.

Stochastic Calculus

Martin Baxter and Andrew Rennie, *Financial Calculus, an Introduction to derivatives pricing*, 1996, Cambridge Univ. Press.

Bernt Oksendal, *Stochastic Differential Equations: an Introduction with Applications*, Springer, 5th ed.

Portfolio Theory

Harry M. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, 1971, Yale Univ. Press.

Reading (cont.)

Beyond Black-Scholes (Stochastic Volatility, Jumps)

T. Bollerslev, R.Y. Chou, and K.F. Kroner, ARCH modeling in Finance, A Review of the Theory and Empirical Evidence, Journal of Econometrics, 1992, 52, 5-59.

S.L. Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, The Review of Financial Studies, 1993, 6, No. 2, 327-343.

Mark S. Joshi, *The Concepts and Practice of Mathematical Finance*, 2003, Cambridge Univ. Press

Alan L. Lewis, *Option Valuation under Stochastic Volatility*: *with Mathematica Code*, Finance Press, 2000. (This is Vol. I). Vol II*: Jumps and Exotics*, Finance Press. (forthcoming summer 2004, along with second printing of Vol. I).

A. L. Lewis, *Fear of Jumps*, Wilmott magazine, Dec. 2002, 60-67.

Example Plot of Call Option Price vs. Stock Price