THE MIXING APPROACH TO STOCHASTIC VOLATILITY AND JUMP MODELS[#]

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March, 2002

1 INTRODUCTION

This article introduces mixing theorems, which offer both a theoretical and computational approach to certain advanced option models. Before explaining them, we first review a little background about option pricing theory. The Black-Scholes-Merton family of models is a well-known and sensible starting framework for understanding option prices. The framework relies on the assumption that the underlying stock price (or security price) follows a process known as geometric Brownian motion (GBM). This model has some very strong points in its favor: (i) it's consistent with stocks as limited liability securities (and so the prices never fall below zero), (ii) it has uncorrelated returns, which are a compelling consequence of highly efficient markets with strong statistical support over many time scales, and (iii) it's very tractable computationally.

However, when you look more closely at GBM, there are problems. For example, the process implies a normal distribution for the logarithmic price returns: $\log(S_t / S_{t-1})$, where the S_t are the stock prices and $(t, t-1)$ is *any* time interval. In practice, say with daily prices, there are too many outliers for this model. This is the "wide-tail" problem. Another problem is that, while actual price returns exhibit very little auto-correlation, the absolute returns $|\log(S_t / S_{t-1})|$ tend to show significant positive auto-correlation, especially at daily or higher frequencies. This is the "volatility-clustering" problem. A third problem is found in the options market: GBM implies the Black-Scholes (BS) formula for option prices. The only input to this celebrated formula that is not strictly observable is the volatility parameter σ (sigma) for the underlying stock. Hence, if you take all the options trading at a given maturity and fit the option prices to the BS model (say using the bid-ask average option price with a simultaneous stock price), then you should recover the unknown constant σ. This parameter is supposed to be a property of the stock price, yet it can be separately fitted for each option at various strike prices *K*. This fitted parameter is called the "implied volatility". In practice, when you do this, the values for σ, in contradiction to the BS model, depend upon the strikes in a rather systematic way: you get a non-constant function

$\sigma(K)$. The graph of this function (or certain close variations) is called the volatility smile or skew or sometimes "smirk" because of its shape. If the BS formula were valid, this graph should be a horizontal (flat) line and the terminology "smile" would not exist. A related problem is that you find that the implied volatility also depends upon the time to expiration.

To solve these problems, practitioners and researchers have explored various alternative theories. If there are more realistic models for the stock price than GBM, then efficient option markets should clearly incorporate them into valuations, at least to the extent that greater realism has a meaningful monetary effect. One natural idea is to make the Black-Scholes' volatility a random process – these are so-called stochastic volatility models. In practical observation, volatility does vary and tends to be mean-reverting. But the details are hard to pin down because volatility is hard to measure, there are probably competing time scales, and parameters may not be stationary. Another natural idea is to allow the stock price to jump (and possibly the volatility to jump, too). A general class of models with jumps is the family of exponential Lévy processes. Jumps are certainly a fact or life for real security prices, although, again, adopting a particular parameterization can be difficult for many of the same reasons already mentioned.

When these approaches are combined in models with both stochastic volatility and jumps, and the various parameters are suitably adjusted, you can obtain a *much* better fit to both actual stock price distributions (with their wide tails), and smile patterns in the options markets. One price you pay for this realism is that it's harder to get from the model to the option price. Mixing theorems can help with this computational problem. Another price you pay is that, unlike the BS theory, the risk attitudes of investors influence option prices. In some cases, this "risk-adjustment" problem can be subsumed under parameter adjustments: that is how we will treat it here in order to stick to one subject.

What mixing theorems do is express the option prices in the more complicated models as a weighted sum of the option prices in simpler *base* models. There are lots of variations on this idea. For example, first consider stochastic volatility models with no jumps. There is a basic mixing theorem which expresses option prices under stochastic volatility as a weighted sum of constant volatility prices. Of course, constant volatility is just the Black-Scholes model, so we are able to express put and call option prices as a weighted sum of BS prices. This mixing idea was first demonstrated, in the special case of no correlation between the stock price and volatility changes, by Hull and White (1987). Then, Romano and Touzi (1997) extended this to the case where the stock price changes and volatility changes are correlated – correlation is very important in understanding options on broad-based indexes, such the S&P500. Romano and Touzi's extension only applied to put and call options, but in Lewis (2000), the theorems were further extended to handle (i) generalized payoff functions (not just puts/calls) and (ii) generalized stock price volatility coefficients $\sigma(S_t)$. Most of the subsequent discussion in this article is a less technical version of material in Chapter 5 of Lewis (2000). But, in addition, we discuss for the first time in this article the extension of mixing to jump processes.

To do that, let's consider models with both stochastic volatility and stock price jumps. First, consider GBM plus stock price jumps (but the diffusion volatility is constant). One version of this is the so-called "jump-diffusion" model created by R.C. Merton (1976) in which the stock price follows GBM most of the time, but can also occasionally have a discontinuous move which is described by a compound Poisson process. In this model, jumps occur randomly, with a certain average frequency. When a jump occurs, the logarithmic price jump is drawn independently from a normal distribution with two parameters describing the mean jump size and jump size volatility. As it turns out, this model can be fairly easily analyzed and has an exact solution which is not much harder to work with than the basic Black-Scholes model. Finally, you can then generalize the jump-diffusion model by making random the continuous volatility parameter – this final model has both stock price jumps and stochastic volatility. For most volatility process specifications, there is no known closed-form option solution. However, a mixing theorem can again be derived which express the solution, under stochastic volatility, as a weighted sum of the Merton jump-diffusion solution. This is developed in Sec. 6 below.

Mixing theorem solutions are often rather formal (as you will see below), so their benefit is not that they magically solve an otherwise intractable problem. Nevertheless, they provide a way of representing a complex solution that has both theoretical and computational advantages. In this article, we will illustrate both. The theoretical application shows how mixing leads to a straightforward proof that the option smile is "symmetrical" about the "at-the-money" strike price in certain stochastic volatility models. The most significant computational application is to improve Monte Carlo techniques. Specifically, mixing theorems *greatly* increase the efficiency of Monte Carlo evaluation of option prices in both basic stochastic volatility models and models with stochastic volatility plus jumps. Because of the wealth of potential applications, I believe mixing ideas are under-appreciated. So, one goal of this article is to emphasize their flexibility and power in taming some of the challenges in mathematical finance.

2 The Basic Mixing Solution for Stochastic Volatility Models

In this section we want to consider stochastic volatility models in a continuous-time world with perfect security markets. Perfect markets have no transaction costs or other frictions and no arbitrage possibilities. The models are expressed as stochastic differential equations (SDEs), which describe how the underlying security prices evolve in time. One immediate complication is that, for the purpose of pricing options, one doesn't really care about the "actual" (or sometimes called the "statistical" process). Instead, the main evolution object becomes the so-called "risk-adjusted" or "pricing" process, which differs from the actual one by some transformations of drifts. These drift transformations are a consequence of the "no-arbitrage" assumption. All of our SDEs should be interpreted as risk-adjusted in this manner.

We'll work our way up to the final case of stochastic volatility plus jumps in a series of steps, beginning with the Black-Scholes case and then adding complexity. For the Black-Scholes' model on a non-dividend paying stock, the pricing process SDE is $dS_t = rS_t dt + \sigma_0 S_t dB_t$, which is geometric Brownian motion. In this SDE, the drift rate of dS/S is the riskless interest rate *r,* and the constant volatility σ_0, apart from a time factor, measures the instantaneous standard deviation of returns. The instantaneous variance rate of returns $V_0 = (\sigma_0)^2$ and we shall also call V_0 the volatility, hopefully without confusion. The process dB_t is a Brownian motion process, which can be thought of as the limiting behavior of $z_t(\Delta t)^{1/2}$, where z_t is a standard normal variate drawn independently every Δt.

By standard arguments, the fair value for an option at time $t = 0$ is given by a discounted expectation $\mathbb{E}_0[\cdots]$ over the payoff function. For a call option striking at *K* that expires in *T* periods, where today's stock price is S_0 and today's volatility is V_0, we write the fair value as $c(S_0, V_0, T)$. (Note the small *c*). Of course, the result is the Black-Scholes formula:

(2.1) $$c(S_0, V_0, T) = e^{-rT}\mathbb{E}_0[(S_T - K)^+] = S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

using $d_\pm = \frac{1}{\sqrt{V_0 T}}\left[\ln\left(\frac{S_0}{Ke^{-rT}}\right) \pm \frac{1}{2}V_0 T\right]$, $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$, and $(x)^+ = \max(0, x)$.

Next, let's consider the modest generalization where the volatility can vary, but in a deterministic manner. For example, suppose the volatility follows an ordinary differential equation (ODE) $dV/dt = \omega - \theta V$, where $V(t=0) = V_0$ and where (ω, θ) are two constants. This ODE is very easy to solve and the answer is $V(t) = \omega/\theta + (V_0 - \omega/\theta)\exp(-\theta t)$. More generally, suppose that we are just given a function $V(t)$, from whatever source, that describes the deterministic volatility evolution. We still want to value a call option at $t = 0$, when the volatility has the value V_0 and there are T periods to expiration. To distinguish this case from the Black-Scholes formula above, we will capitalize the new formula: $C(S_0,V_0,T)$. As shown by Merton in his classic 1973 paper "Theory of Rational Option Pricing", by a time change argument, one discovers that

(2.2) $$C(S_0,V_0,T) = c(S_0,V^{\mathit{eff}},T) \text{ , where } V^{\mathit{eff}} = \frac{1}{T}\int_0^T V(s)ds \, .$$

In words, formula (2.2) says that under deterministic volatility, we can continue to value options using the Black-Scholes formula, but we have to use an effective volatility. Moreover, the effective volatility is just the time-average of the deterministic volatility. For example, our simple ODE solution above is easily integrated to yield

$$V^{\mathit{eff}} = \frac{\omega}{\theta} - \left(\frac{V_0}{\theta} - \frac{\omega}{\theta^2}\right)\frac{\exp(-\theta T) - 1}{T},$$

which is then substituted into $c(S_0,V^{\mathit{eff}},T)$. If we were given an *arbitrary* ODE for the volatility, say of the homogeneous form $dV = b(V)dt$, then we could imagine solving this for $V(t)$ such that $V(t=0) = V_0$. Then, again we would use (2.2) to get the call option value.

Now we are ready to turn to the case of stochastic volatility. Instead of $dV = b(V)dt$, we add a random (noisy) component; hence $dV_t = b(V_t)\,dt + a(V_t)\,dW_t$. We have introduced a new source of uncertainty dW_t, another Brownian motion, which may be correlated with the Brownian motion dB_t that drives the stock price. For example, it is common to observe in broad-based indexes, like the S&P500, that when prices fall abruptly, volatility usually rises and vice-versa. This "leverage effect" is generated by a negative correlation ρ, with typical estimates in the range $\rho \approx -0.5$ to -0.8 for this particular index.

As we mentioned above, there are drift transformations associated with the absence of arbitrage, so our volatility SDE is "risk-adjusted". That is, the volatility drift $b(V_t)$ can differ from the actual volatility drift because of investor risk attitudes. If we write out both the stock price SDE and the volatility SDE together, our stochastic volatility system becomes

(2.3) $$\begin{cases} dS_t = rS_t dt + \sigma_t S_t dB_t \\ dV_t = b(V_t)\,dt + a(V_t)\,dW_t \end{cases}$$

Remember that we said that the two Brownian motions are correlated? We can express this in an explicit way by writing

(2.4) $$dB_t = \rho_t\,dW_t + (1-\rho_t^{\,2})^{1/2} dZ_t,$$

where dZ_t is now another Brownian motion that is independent (hence, uncorrelated) with the noise dW_t. (We allow the correlation ρ_t to depend, at most, on the volatility V_t, but *not* the stock price or explicit time). If you insert (2.4) into (2.3), then the only Brownian motions that appear will be the dW_t and the dZ_t, which are independent. Since we want to create a Monte Carlo procedure, its helpful to think of (2.3) as the $\Delta t \to 0$ limit of a discrete-time process, where $t = 0, \Delta t, \cdots, T$. In discrete-time, we can simulate the SDE by drawing two independent standard normal variates $\hat{W}_t, \hat{Z}_t$ at each time step $0 \le t \le T - \Delta t$. (Some notation: the hat, ^, distinguishes these discrete-time random variables from Brownian motion processes. For the other variables, it should not cause confusion if we use the same notation for their discrete-time counterparts). So, a Monte Carlo version of our system is

$$\text{(2.5)} \qquad \begin{cases} \Delta S_t = rS_t \Delta t + \sigma_t S_t \left[\rho_t \hat{W}_t + (1-\rho_t{}^2)^{1/2} \hat{Z}_t \right] \sqrt{\Delta t} \\ \Delta V_t = b(V_t)\Delta t + a(V_t)\hat{W}_t \sqrt{\Delta t} \end{cases}$$

The call option price is the limiting value of the discounted expectation of the payoff, as $\Delta t \to 0$. This is both the limit of a Monte Carlo average, and also just a multiple integral over Gaussian distributions. That is, we have the formula

$$\text{(2.6)} \qquad \begin{aligned} C(S_0, V_0, T) &= e^{-rT} \mathbb{E}_0[(S_T - K)^+] \\ &= \lim_{\Delta t \to 0} e^{-rT} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (S_T - K)^+ \prod_{t=0}^{T-\Delta t} \exp\left[-\tfrac{1}{2}(\hat{Z}_t^2 + \hat{W}_t{}^2) \right] \frac{d\hat{Z}_t d\hat{W}_t}{2\pi} . \end{aligned}$$

In (2.6), think of S_T as a complicated function of each particular sequence of the integration variables from $t = 0$ to $t = T - \Delta t$. Indeed, we will actually write down useful formulas for this function. Here's how.

First, imagine that we have already made a complete sequence of drawings of the $\hat{W}_t$ for $t = 0, \Delta t, 2\Delta t, \cdots$. Then, we can use this sequence to determine the volatility V_t at each time step; to do so, just evaluate, for $t = \Delta t, 2\Delta t, \cdots, T$,

$$\text{(2.7)} \qquad V_t = V_0 + \sum_{s=0}^{t-\Delta t} b(V_s)\Delta t + \sum_{s=0}^{t-\Delta t} a(V_s)\hat{W}_s \sqrt{\Delta t} .$$

Of course, since we know V_t, we also know σ_t at each step. Now, *given* this sequence of σ_t, we can now imagine doing the sequence of drawings of the $\hat{Z}_t$. Once we have those values, we can write down the value for the terminal stock price S_T. This is the solution to the first equation in (2.5). With a little algebra, it can be shown to be given by

$$S_T = S_0\, e^{Y_T} \exp\left(rT - \tfrac{1}{2} \sum_{t=0}^{T-\Delta t} (1-\rho_t^2)\sigma_t^2 \Delta t + \sum_{t=0}^{T-\Delta t} (1-\rho_t^2)^{1/2} \sigma_t \hat{Z}_t \sqrt{\Delta t} \right), \tag{2.8}$$

where

$$Y_T = -\tfrac{1}{2} \sum_{t=0}^{T-\Delta t} \rho_t^2\, \sigma_t^2 \Delta t + \sum_{t=0}^{T-\Delta t} \rho_t \sigma_t \hat{W}_t \sqrt{\Delta t}\,. \tag{2.9}$$

Now you may not recognize it immediately, but a little reflection shows that (2.8) is, as $\Delta t \to 0$, the solution to a *deterministic volatility*, Black-Scholes SDE: $dS_t = rS_t dt + \sigma_t^{eff} S_t dZ_t$. The solution to this SDE, which (i) starts at $S_0\, e^{Y_T}$ instead of the usual S_0, and (ii) has an effective volatility $\sigma_t^{eff} = (1-\rho_t^2)^{1/2} \sigma_t$, is given by (2.8). This observation implies that we can interpret the entire set of integrations in (2.6) over the $d\hat{Z}_t$ variables, *conditional* on holding the $\hat{W}_t$ fixed, as a deterministic volatility problem with the two modifications that we have just explained.

But, the expectation of $(S_T - K)^+$ has Merton's simple solution under deterministic volatility, as we explained above. Namely, just use the B-S formula, where the variance parameter is replaced by an effective variance $V \to V^{eff} = \int_0^T V_t\, dt / T$. Also, the stock price adjustment is just a multiplicative adjustment to today's stock price in the same B-S formula. In other words, moving back to continuous time again, define the effective stock price and effective volatility by

$$S_T^{eff} = \lim_{\Delta t \to 0} S_0 e^{Y_T} = S_0 \exp\left(-\tfrac{1}{2} \int_0^T \rho_t^2 \sigma_t^2 dt + \int_0^T \rho_t \sigma_t dW_t \right), \tag{2.10}$$

$$V_T^{eff} = \lim_{\Delta t \to 0} \frac{1}{T} \sum_{t=0}^{T-\Delta t} (1-\rho_t^2) \sigma_t^2 \Delta t = \frac{1}{T} \int_0^T (1-\rho_t^2) \sigma_t^2 dt\,. \tag{2.11}$$

Also, let's introduce a simple bracket notation $\langle \cdots \rangle$ for the remaining integrations over the volatility process in (2.6). That is, the bracket indicates an expectation over the multivariate Gaussian associated with the all the $\{dW_t\}$ variables, which are the ones driving the volatility. Then, we have shown that the call option value under stochastic volatility is given by

$$\left\langle c(S_T^{eff}, V_T^{eff}, T) \right\rangle = \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} c(S_T^{eff}, V_T^{eff}, T) \prod_{t=0}^{T-\Delta t} \exp\left(-\tfrac{1}{2} \hat{W}_t^2 \right) \frac{d\hat{W}_t}{(2\pi)^{1/2}} \tag{2.12}$$

In summary, we have argued for the validity of the following theorem:

MIXING THEOREM (Romano and Touzi, 1997): *Let* $C(S_0,V_0,T)$ *be the call option price under the risk-adjusted, stochastic volatility process of* (*2.3*). *Let* $c(S_0,V_0,T)$ *be the Black-Scholes formula of* (*2.1*). *And, let the effective stock price* S_T^{eff} *and the effective volatility* V_T^{eff} *be given by* (*2.10*) *and* (*2.11*) *respectively. Then, using the bracket notation of* (2.12),

$$C(S_0,V_0,T) = \left\langle c(S_T^{eff}, V_T^{eff}, T) \right\rangle . \tag{2.13}$$

In words again, the option value under stochastic volatility is a weighted sum or mixture of the Black-Scholes values with an effective stock price and effective volatility. The effective variables depend *only* upon the volatility process. Hence, the problem reduces to a pricing expectation over the risk-adjusted volatility process alone.

Zero correlation. In general, $\left\langle S^{eff} \right\rangle = S_0$, but when $\rho_t = 0$, then $S^{eff} = S_0$. In that case, introduce $P(U_T;V_0,T)$, the probability distribution of the integrated volatility $U_T = \int_0^T V_t dt$. With that distribution, (2.13) can be interpreted as

$$C(S_0,V_0,T) = \int_0^\infty c\left(S_0, \frac{U_T}{T}, T\right) P(U_T;V_0,T)\, dU_T \tag{2.14}$$

Hull and White (1987) established this case.

3 Closed-form Examples

In general, don't try too hard to solve mixing problems in closed-form. Nevertheless, there are some relatively simple cases that help clarify the rather formal relationships discussed above. The first example makes use of a function we call the fundamental transform. All you really need to know about this function is two things: (i) it satisfies a certain partial differential equation (PDE), which will be explained in the example, and (ii) once you have it, you can get the probability distribution $P(U_T;V_0,T)$ needed for (2.14) pretty easily. (We use (2.14) because the simplest examples have $\rho = 0$)

Example 3.1. Volatility as a Square Root Process

One of the easiest mixing theorem examples uses the square-root model with no drift. In this example, we take $dV_t = \xi \sqrt{V_t}\, dW_t$. With that, the fundamental transform $H(c,V,T)$ satisfies the PDE $H_T = (1/2)\xi^2 V H_{VV} - cVH$, where the subscripts indicate derivatives. In addition, the fundamental transform satisfies the initial condition $H(c,V,T=0)=1$. The *meaning* of the fundamental transform is discussed briefly below.

Take $\xi=1$; it can be shown that the solution to the PDE we just introduced is $H(c,V,T)=\exp\{-V(2c)^{1/2}\tanh[(2c)^{1/2}T/2]\}$. Now what is the relation between the fundamental transform and $P(U_T;V_0,T)$? It turns out that the fundamental transform is the *characteristic function* of the integrated variance density: $H(c)=\int_0^\infty e^{-cU}P(U)\,dU$, suppressing common arguments. This integral is also a *Laplace transform*. Hence, given $H(c,V,T)$, we can obtain $P(U;V_0,T)$, needed for (2.14), from the Laplace transform inversion formula:

$$P(U;V_0,T)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{cU}H(c,V_0,T)\,dc, \tag{3.1}$$

where the integral runs along a vertical line in the complex *c*-plane to the right of any singularities. In our case, we can take $\gamma=0$; i.e. integrate along the imaginary *c*-axis . We'll do this by letting $c=i\,y$ and, for definiteness take $V_0=T=1$, letting $P(U)=P(U;1,1)$.

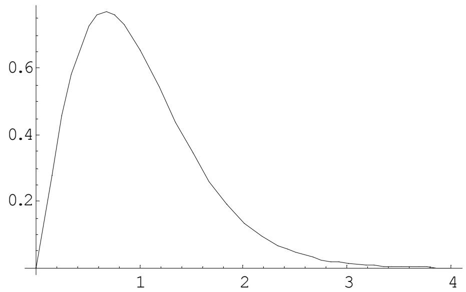
This may all sound complicated, but it's really easy to implement in a symbolic programming language such as Mathematica. For example, here is the code – it's just one line – where the dy integration is cut off at a maximum value **`ymax`**:

```
P[U_,ymax_]:= N[1/Pi *
              NIntegrate[Re[E^(-Sqrt[2 I y] Tanh[Sqrt[2 I y]/2]) E^(I y U)],

                                                {y,0,ymax}, MaxRecursion->20]]
```

Most of the syntax will probably make sense, even if you have never used Mathematica. (Numerical integrations are performed by the built-in function **`NIntegrate[...]`**). By plotting the result, we can see what the density function of the integrated variance looks like:

```
In[93]:= Timing[Plot[P[U, 300], {U, 0, 4}, PlotDivision -> 1]]
```




```
Out[93]= {12.14 Second, - Graphics -}
```

Note that the density vanishes as $U \to 0$; this should be a general feature of *any* stochastic volatility model. As $U \to \infty$, the density for the example vanishes faster than any power of U; this is a consequence of the fact that the fundamental transform $H(c,V,T) = \exp\{-V(2c)^{1/2}\tanh[(2c)^{1/2}T/2]\}$ is analytic in c near $c = 0$. Here's a short proof: the Taylor series for the hyperbolic tangent function, $\tanh x$, about $x = 0$, only contains positive odd powers of x. Hence, $H = 1 + a_1 c + a_2 c^2 + \cdots$ with finite coefficients a_i. So every c-derivatives of $H(c,V,T)$ exists at $c = 0$. It's a well-known fact and you can see it from $H(c) = \int_0^\infty e^{-cU} P(U)\,dU$, that derivatives of the characteristic function of a density, in our case $H(c)$, generate moments $< U^m >$, $m = 0,1,2,\cdots$ Each moments is finite, which can be true only if $P(U)$ vanishes faster than any power of U as $U \to \infty$.

To complete the example, we do another numerical integration to evaluated the call option value using the mixing theorem (2.14). With $S_0 = K = 100$, where K is the strike price (and remember that $V_0 = T = 1$), the final call option value is 36.48. This is correct and can be confirmed by other means.

The example shows that everything works out correctly and gives you a general picture of what the density for the integrated volatility looks like.

Example 3.2. Volatility as Geometric Brownian Motion
The simplest case here is to drop all drifts and consider the risk-adjusted process

$$\begin{cases} dS_t = \sigma_t S_t \, dZ_t \\ dV_t = \xi V_t \, dW_t \end{cases},$$

where $V_t = \sigma_t^2$, ξ is a constant, and the two Brownian motions are independent. Let $C(S_0, K, V_0, T)$ be the value of a call option striking at *K* with *T* periods to expiration. While this model can be solved for any value of *T,* it is especially simple in the limit where $T \to \infty$. It can be shown that the integrated volatility density of (2.14) is given by

$$P(U;V_0,T) \approx \frac{2V_0}{U^2\xi^2}\exp\left(-\frac{2V_0}{U\xi^2}\right) = P_\infty(U;V_0), \qquad \text{as} \quad T \to \infty. \tag{3.2}$$

Then, from the mixing theorem (2.14) , again as $T \to \infty$, we have

(3.3) $\quad C(S_0,K,V_0,T) \to C_\infty(S_0,K,V_0) =$

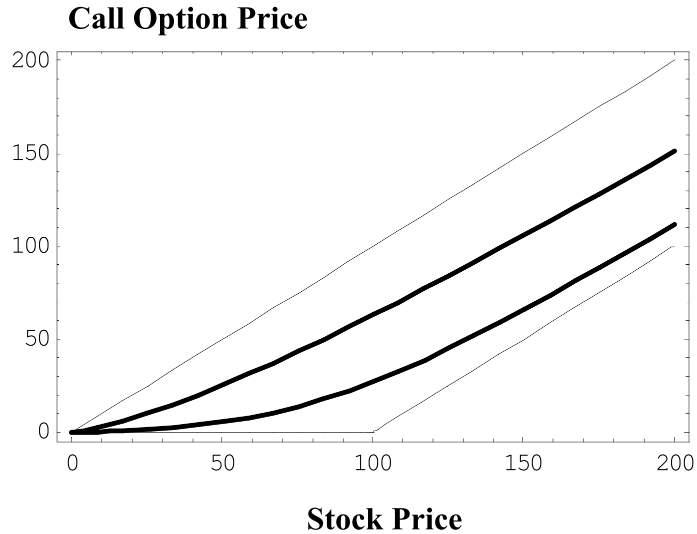
$$\frac{2V_0}{\xi^2}\int_0^\infty \left\{ S_0\Phi\left(\frac{[Log(S_0/K)+U/2]}{\sqrt{U}}\right) - K\Phi\left(\frac{[Log(S_0/K)-U/2]}{\sqrt{U}}\right)\right\} \exp\left(-\frac{2V_0}{U\xi^2}\right)\frac{dU}{U^2}$$

$$= S_0 - \sqrt{S_0K}\exp\left(-\left[\frac{1}{4}Log^2\left(\frac{S_0}{K}\right)+\frac{V_0}{\xi^2}\right]^{1/2}\right).$$

Note that $C_\infty(S_0,K,V_0)$ is strictly less than the stock price. In the B-S model, as the time to expiration grows large, the option price becomes the stock price for any non-negative interest rate and positive volatility. The new behavior of (3.3) is caused by the volatility drift toward the origin. Fig. 3.1 below plots the call value in (3.3) versus the stock price with $K = 100$ and $V_0/\xi^2 = 0.1$ (lower bold curve) and $V_0/\xi^2 = 1$ (upper bold curve).

In the example, the B-S implied volatility at $T = \infty$ is zero. If you developed the B-S implied volatility versus the time to expiration *T*, you would expect to find it eventually decreasing with *T,* since it's heading to zero. This effect was seen in Monte Carlo studies of this model by Hull and White (1987) and explains results they found surprising.

Fig. 3.1 Call Price under Stochastic Volatility as $T \to \infty$. Volatility Process is Geometric Brownian Motion



4 Monte Carlo Mixing

The discrete-time version of the mixing theorem yields a simple Monte Carlo procedure, requiring only the draw of a single normal variate at each time step. This is probably the most useful application. We discuss some of the details in this section.

The bracket notation $\langle \cdots \rangle$ that we introduced at (2.12) can be re-used with a slightly different interpretation. In this section, we let $\langle \cdots \rangle$ mean an average over N Monte Carlo (MC) simulations with time-step Δt. Then, the mixing theorem derivation establishes a MC pricing formula. Instead of the call option, let's switch to a put option. The mixing theorem is

$$P(S_0, V_0, T) = \lim_{\substack{N \to \infty \\ \Delta t \to 0}} \left\langle p(S^{eff}, V^{eff}, T) \right\rangle. \tag{4.1}$$

We emphasize the put option to stress that the MC statistics are often much better for a put option than a call. This is especially true in large volatility limits or in other difficult cases. You can always recover call option prices from the put-call parity formula, rather than direct MC averaging.

To implement (4.1), draw a *single* standard normal variate $\hat{Z}_t$ at each time step, $t = 0, \Delta t, \ldots, T - \Delta t$. Except at the boundaries, this random draw is used to update the sequences

(4.2) $$Y_{t+\Delta t} = Y_t - \tfrac{1}{2}\rho_t^2 \sigma_t^2 \Delta t + \rho_t \sigma_t \hat{Z}_t \sqrt{\Delta t} \ ,$$

(4.3) $$V_{t+\Delta t} = V_t + b(V_t)\Delta t + a(V_t)\hat{Z}_t \sqrt{\Delta t} \ ,$$

We start with $Y_0 = 0$ and the given $V_0 > 0$. Typical volatility models can take on any non-negative value. For the simulation, (i) if the volatility origin is crossed, reflecting the process back to positive values is often correct, and (ii) if the volatility can explode (a possibility after risk-adjustment), simply introduce a large upper bound cutoff.

Then, the result of a single simulation run is calculated from the B-S formula with the arguments

$$S^{eff} = S_0 \exp(Y_T) \ \text{ and } V^{eff} = \frac{1}{T}\sum_{t=0}^{T-\Delta t} (1-\rho_t^2) V_t \Delta t \,.$$

The exact continuous-time result is the limiting average (4.1).

An example. To see the performance of the method, we created a short C-code program, which implements this procedure. For variance reduction, the program uses both $\hat{Z}_t$ and $-\hat{Z}_t$ for each single simulation; this is the well-known antithetic technique. The volatility follows the GARCH diffusion, which is given by the SDE

$$dV_t = (\omega - \theta V_t)\, dt + \xi V dW_t$$

Table 4.1 entries show the MC put price, MC standard error in parenthesis, and the Black-Scholes implied volatility (σ^{imp}, in percent, annualized). Entries are for various strike prices and stock-volatility correlations ρ. The model parameters are $S = 100$, $\omega_a = 0.09$, $\theta_a = 4$, $\xi_a = 1$, where the subscript emphasizes *annualized* units. With 250 days-per-year, we took $\Delta t_a = 1/250$ and $T_a = 20/250$ years (20 days to expiration). The example also takes $r = 0$.

The last line of each row in the table shows typical smile patterns, where, for example, out-of-the-money put prices are higher than B-S prices under a negative correlation. Two of the rows are plotted in Fig. 4.2 (with interpolation).

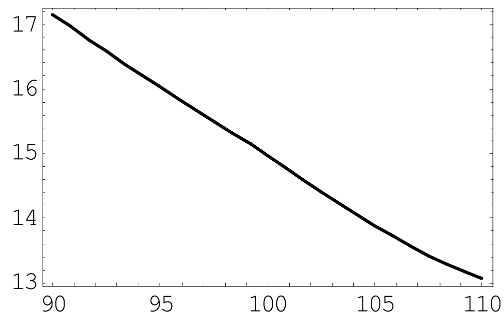
Table entries are based on 100,000 simulation runs; one can see that the MC standard errors are all less than 1 penny, and some significantly less (see the row with $\rho = 0$). Since each simulation run feeds its results to the Black-Scholes model, it's much more efficient than a *standard* MC simulation implementing (2.5). The standard MC would require two Gaussian draws at each step and would use the payoff function at expiration, *not* the BS formula. Since the MC mixing version is so easily coded, the method is a very effective way to get fairly accurate prices for short-term options without much hassle.

Table 4.1 Monte Carlo Put Prices using the Mixing Theorem.

Correlation	**Strike Price**				
ρ	**90**	**95**	**100**	**105**	**110**
-1.0	0.0245	0.285	1.689	5.207	10.006
	(0.0006)	(0.002)	(0.004)	(0.002)	(0.0009)
	17.16	17.03	14.97	13.89	13.06
-0.50	0.0161	0.257	1.688	5.239	10.012
	(0.0001)	(0.0007)	(0.001)	(0.0008)	(0.0005)
	17.22	15.54	14.96	14.47	14.15
0.0	0.0095	0.229	1.688	5.272	10.021
	(7×10^{-6})	(3×10^{-5})	(3×10^{-5})	(3×10^{-5})	(1×10^{-5})
	15.19	15.02	14.96	15.01	15.15
0.50	0.0046	0.200	1.689	5.302	10.031
	(1×10^{-5})	(0.0003)	(0.0006)	(0.0002)	(0.0005)
	14.02	14.46	14.97	15.52	17.05
1.0	0.0015	0.170	1.692	5.332	10.043
	(0.0001)	(0.0014)	(0.003)	(0.001)	(0.0012)
	12.63	13.85	15.00	15.99	17.89
BS values:	0.0085	0.228	1.692	5.270	10.019
	15.00	15.00	15.00	15.00	15.00

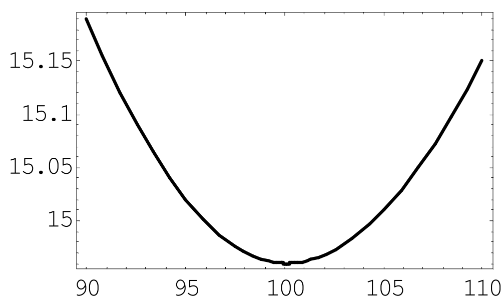
Fig. 4.2 Smile Patterns for the GARCH Diffusion Monte Carlo Method using the Mixing Theorem

Implied Volatility (ρ = -1)



Strike Price

Implied Volatility (ρ = 0)



Strike Price

5 Symmetric Smiles

Notice from Fig. 4.2, that when the correlation $\rho = 0$, we seem to have an almost symmetric shape about the strike $K = 100$. It would be exactly symmetric if we had used a slightly different measure of the "moneyness". The exact result, due to Renault and Touzi (1996, Proposition 3.1), establishes that the smile is symmetric as a function of the moneyness variable $X = \ln(S_0 / K) + rT$. In this section, we show their argument, which is based on mixing.

Zero correlation is typical of currency options, where the symmetry between the two currencies being exchanged argues against the existence of a leverage effect. A symmetric smile is also seen in some commodity options. Written out more explicitly, (2.14) reads

(5.1) $$C(S_0,V_0,T) = S_0 \int_0^\infty g(X,U_T) P(U_T;V_0,T)\,dU_T ,$$

where $$g(X,U_T) = \Phi\left(\frac{X}{\sqrt{U_T}} + \frac{1}{2}\sqrt{U_T}\right) - e^{-X}\Phi\left(\frac{X}{\sqrt{U_T}} - \frac{1}{2}\sqrt{U_T}\right).$$

It's easy to verify the property:

(5.2) $$g(-X,U) = e^X g(X,U) + 1 - e^X .$$

Hence, if you define $f(X,V,T)$ by $C(S,V,T) = S\, f(X,V,T)$, then (5.2) implies that $f(X,V,T)$ inherits this same property:

(5.3) $$f(-X,V,T) = e^X f(X,V,T) + 1 - e^X .$$

In particular, (5.3) holds under constant volatility, in which case we write $f_{BS}(X,V,T)$. The implied volatility $V^{imp}(X,V,T)$ is the solution to $f(X,V,T) = f_{BS}\left(X, V^{imp}(X,V,T),T\right)$. Hence using (5.3) twice:

$$\begin{aligned} f(-X,V,T) &= f_{BS}\left(-X, V^{imp}(-X,V,T),T\right) \\ &= e^X f_{BS}\left(X, V^{imp}(-X,V,T),T\right) + 1 - e^X \\ &= e^X f\left(X,V,T\right) + 1 - e^X . \end{aligned}$$

The last two equations imply that

(5.4) $$f_{BS}\left(X, V^{imp}(-X,V,T),T\right) = f(X,V,T) = f_{BS}\left(X, V^{imp}(X,V,T),T\right).$$

Since $f_{BS}(X,V,T)$ is single-valued as a function of V, the two expressions using f_{BS} in (5.4) can only be equal if

(5.5) $$V^{imp}(-X,V,T) = V^{imp}(X,V,T) \quad (\rho = 0)$$ ■

6 Adding Stock Price Jumps

Many researchers believe that the marketplace option patterns such as the smile or skew are best explained with some combination of stochastic volatility and jumps. It turns out to be relatively easy to generalize our mixing results to a model that adds stock price jumps to the system (2.3). For example, suppose we add the independent, log-normal, stock price jump process of Merton (1976); then (2.3) becomes

$$(6.1) \qquad \begin{cases} dS_t = (r - \lambda k) S_t dt + \sigma_t S_t dB_t + S_t dQ_t \\ dV_t = b(V_t)\, dt + a(V_t)\, dW_t \end{cases}.$$

The jumps are represented by dQ_t, a symbol for an independent compound Poisson process, with intensity λ and jump amplitude $e^x - 1$, where $x \sim N(\mu_J, \sigma_J^2)$. ($x$ is normally distributed with the mean and variance shown). In other words, when the stock price jumps, we have $S_{t^-} \to S_t e^x$, where t^- is the time just before the jump.

The constant $k = \exp(\mu_J + \sigma_J^2 / 2) - 1$, and its appearance in the drift keeps the expected stock price change equal to $r\,dt$, as it must be, in a risk-adjusted world.

Now you can repeat the arguments of Section 2, except that instead of taking the *base* model to be the Black-Scholes model, the base model is the option price under the evolution

$$(6.2) \qquad dS_t = (r - \lambda k) S_t dt + \sigma_0 S_t dB_t + S_t dQ_t,$$

which again has constant (diffusion) volatility. Getting option prices from (6.2) was solved by Merton in his 1976 paper. His answer was a simple power series using the Black-Scholes formula. We refer the reader to the literature for the specific formula, but we will just write it as $c_M(S_0, V_0, T)$ with the subscript indicating Merton's solution. Then, the mixing theorem becomes, for the process (6.1),

$$(6.3) \qquad C(S_0, V_0, T) = \left\langle c_M(S_T^{eff}, V_T^{eff}, T) \right\rangle,$$

with the effective arguments defined exactly as before. Again, (6.3) is easily implemented by Monte Carlo.

Other types of jump processes are possible: a very flexible class of models that generalizes (6.2) is the exponential Lévy family. There is a simple formulas for the option value under any exponential Lévy process: see Lewis (2001).

The two Monte Carlo mixing theorems discussed here, one with stochastic volatility, and one with stochastic volatility plus jumps, have been implemented in a calculator program available for download at http://www.optioncity.net .

End notes

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