Center for Research in
Financial Mathematics and Statistics (UCSB)
Seminar Series (Feb 26, 2007)

"Geometries and Smile Asymptotics for a Class of Stochastic Volatility Models"

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Overheads to be posted at www.optioncity.net **(Publications)**

Topics (1-3:in literature; 4:new)

1. **What are implied volatility smiles & why asymptotics?**

2. **The Main Theorem for computation at T = 0.**

3. **General approaches to the computation:**
   (i) **Compute geodesics.**
   (ii) **Solve a generalized Eikonal problem.**
   (iii) **Take a limit with a characteristic function.**

4. **Elements of the solution for the CEV(p)-vol model:** $(p \in R, |\rho| < 1)$

$$\begin{cases} dS = \sqrt{V} S \left\{ \rho\, dB(1) + \sqrt{1-\rho^2}\, dB(2) \right\} \\ dV = V^p\, dB(1) \end{cases}$$

Acknowledgments and a few sources

I have greatly benefited from much correspondence with Martin Forde related to today's topic, whom I thank.

I also thank Greg Egan for ideas on visualization/embedding.

Finance related:

Avellaneda, M. "From SABR to Geodesics". 2005 slides

Berestycki, H., J. Busca, and I. Florent. "Computing the Implied Volatility in Stochastic Volatility Models", Comm. Pure & App. Math., 57, No. 10, Oct. 2004, 1352-1373.

Forde, M. "The small-time behavior of diffusion and time-changed diffusion processes on the line", arXiv:math.PR/0609117

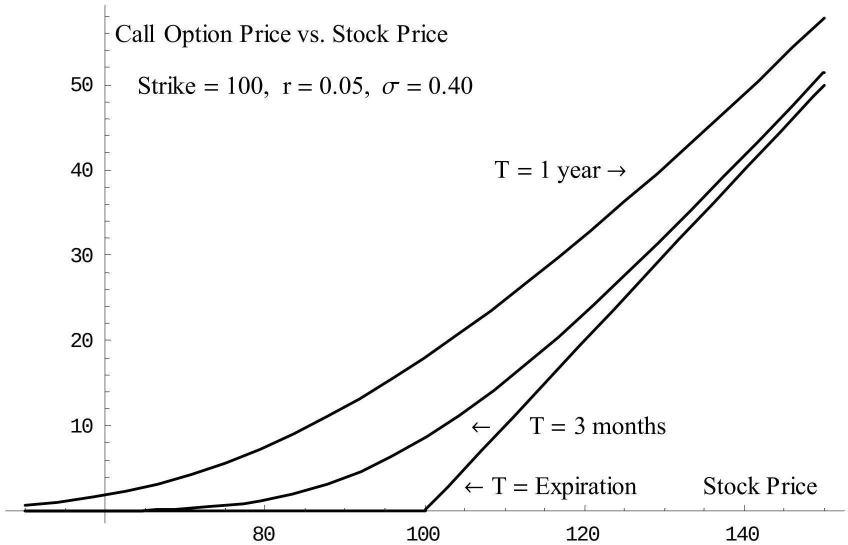
Probability related:

Varadhan, S.R.S., "On the behavior of the Fundamental Solution of the Heat Equation with Variable Coefficients", and "Diffusion Processes in a Small Time Interval". Comm. Pure. Appl. Math. 20, 431-455, and 659-685 (1967)

Riemannian geometry/physics related:

Schutz, B. A First Course in General Relativity, Cambridge Univ. Press, 1990.

Option Prices follow the Black-Scholes model with a "custom volatility" – the implied volatility



<u>Black-Scholes formula</u>:

$$C_{BS}(S,K,T,\sigma) = S\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where $\Phi(z) = \int_{-\infty}^{z} e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ **= cumulative normal**

and $$d_\pm = \frac{\log(S/Ke^{-rT}) \pm \sigma^2 T/2}{\sigma\sqrt{T}}$$

<u>Real-world price</u>: $C_{market} = C_{BS}(S,K,T,\sigma_{implied})$

<u>State-dependent model</u>: $C(S,K,T,\theta) = C_{BS}(S,K,T,\sigma_{implied})$

Of course, for this to work: $\sigma_{implied} = f(S,K,T,\theta)$

Stochastic Volatility Models

<u>Working example</u>: CEV(p)-vol model: ($V = \sigma^2$)

$$dS_t = rS_t dt + \sigma_t S_t dB_t^{(1)}$$
$$dV_t = b(V_t)dt + \xi V_t^p(\rho dB_t^{(1)} + \sqrt{1-\rho^2} dB_t^{(2)})$$

<u>For this class of models:</u>
(Stock price level independent/Translation invariant):

$$\sigma_{implied} = f(T, x, y)$$

where $x = \log(S / K) =$ **log(Stock Price/Strike Price)**

$y = V =$ **Stochastic volatility**

In general, $\sigma_{implied}$ must be numerically computed. But …
The <u>very</u> nice property is that it has a
<u>formal power series</u> (all diffusions):

$$(*) \quad \sigma_{implied} = f^{(0)}(x,y) + T\ f^{(1)}(x,y) + T^2\ f^{(2)}(x,y) + \cdots$$

Proof: (I) Substitute $C(S,K,T,\theta) = C_{BS}(S,K,T,\sigma_{implied})$

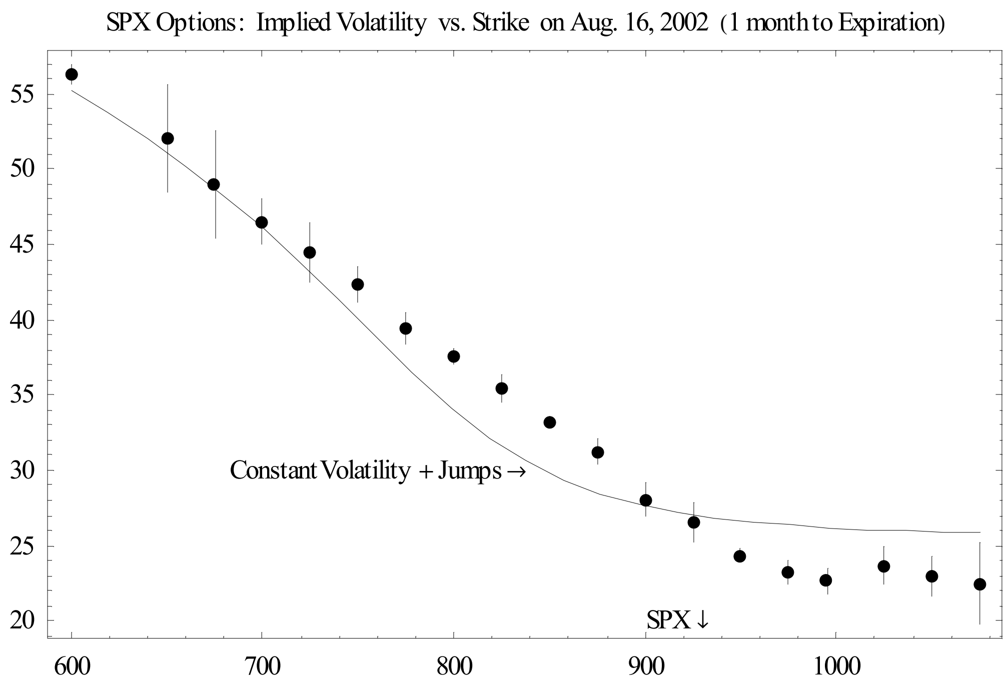
in the PDE for $C(S,K,T,\theta)$ (generic n-factor diffusion)

(II) Result is ugly, but ansatz (*) *works* (ugly$\Rightarrow$beautiful)

Typical market example (SPX)

$$\sigma_{imp}(T = \frac{1}{12}\ \mathrm{yr}, x, y \approx \frac{25\%}{\mathrm{yr}})$$

SPX implied volatility $\sigma_{imp}(x)$, one month-to-go



Today, we explain how to compute the leading

$T \to 0$ behavior in stochastic volatility models:

$$\sigma_{imp}(x, y) \triangleq f^{(0)}(x, y) = \lim_{T\to 0} \sigma_{implied}(T, x, y)$$

The Main Theorem for computing Asymptotic Smile $\sigma_{imp}(x,y) = \lim_{T\to 0}\sigma_{implied}(T,x,y)$

Before stating it, we need a lemma (real proofs: see Varadhan):
Background: the call option price is determined by

$$C(T,S_0,V_0;K) = e^{-rT}E_{(S_0,V_0)}[(S_T-K)^+]$$
$$= e^{-rT}\int_0^{S_T}\max[0,S_T-K]\,q(T,S_0,V_0;S_T)dS_T$$

where the probability transition density

$$q(T,S_0,V_0;S_T)dS_T = \mathrm{P}_{(S_0,V_0)}[S_T\in dS_T].$$

reflects arriving at the terminal stock price S_T with <u>any</u> volatility.
This is distinguished from the 'complete' transition density:

$$p(T,S_0,V_0;S_T,V_T)dS_TdV_T = \mathrm{P}_{(S_0,V_0)}[S_T\in dS_T,V_T\in dV_T]$$

Let's abbreviate the 'state variables' by $\vec{x}_t = (S_t,V_t)$.
By the Markov property, for any time sub-division $T = n\Delta t$,

$$q(T,S_0,V_0;S_T) =$$
$$\int p(\Delta t,\vec{x}_0;\vec{x}_{t_1})p(\Delta t,\vec{x}_{t_1};\vec{x}_{t_2})\cdots p(\Delta t,\vec{x}_{t_{n-1}};\vec{x}_{t_n})\,dx_{t_1}\cdots dx_{t_{n-1}}dV_T$$

The Main Theorem (cont)

$$q(T, S_0, V_0; S_T) =$$
$$\int p(\Delta t, \vec{x}_0; \vec{x}_{t_1}) p(\Delta t, \vec{x}_{t_1}; \vec{x}_{t_2}) \cdots p(\Delta t, \vec{x}_{t_{n-1}}; \vec{x}_{t_n})\, dx_{t_1} \cdots dx_{t_{n-1}} dV_T$$

<u>Heuristic argument</u>:

$p(\Delta t, \vec{x}; \vec{y})$ is the transition density for a $[2D]$ diffusion process with drift $\vec{b}_t \equiv \vec{b}(\vec{x}_t)$ and variance-covariance matrix

$$a_t = [a_{ij}(\vec{x}_t)], \quad (i, j = 1, \cdots, D)$$

For small enough Δt, the transition densities must be approximately D-dimensional Gaussian:

$$p(\Delta t, \vec{x}; \vec{y}) \approx \frac{1}{(2\pi)^{D/2} (\det a)^{1/2}} \times$$
$$\exp\left\{ -\frac{1}{2\Delta t} [\vec{y} - \vec{x} - \vec{b}(x)\Delta t]' a^{-1}(x) [\vec{y} - \vec{x} - \vec{b}(x)\Delta t] \right\}$$

To leading order, the drifts $\vec{b}(x)\Delta t$ don't contribute:

$$q(T, S_0, V_0; S_T) \approx$$
$$\int \exp\left\{ -\frac{1}{2\Delta t} \sum_{i=1}^{n} (x_{t_i} - x_{t_{i-1}})' a^{-1}(x_{t_{i-1}})(x_{t_i} - x_{t_{i-1}}) \right\} dx_{t_1} \cdots dx_{t_{n-1}} dV_T$$

Note: I am writing $x_t = D$-vector with no arrows now [D = 2].

The Main Theorem (cont)

$$q(T,S_0,V_0;S_T) \approx$$
$$\int \exp\left\{-\frac{1}{2\Delta t}\sum_{i=1}^{n}(x_{t_i}-x_{t_{i-1}})'a^{-1}(x_{t_{i-1}})(x_{t_i}-x_{t_{i-1}})\right\} dx_{t_1}\cdots dx_{t_{n-1}}dV_T$$

In the limit, the points $\{x_{t_i}\} \to \{x_t\}$ create a continuous path (for <u>any</u> diffusion). The integrand is a maximum along the paths $\{x_t\}$ which minimize the sum and becomes concentrated there (saddle point/steepest descent/WKB/etc idea). Interpret

$$g \equiv a^{-1}(x) = [g_{ij}(x)] \text{ as a metric tensor}$$

With <u>implied sums</u> $(i, j = 1,\cdots,D)$ on upper/lower repeated indices:

$$\frac{1}{2\Delta t}\sum_{i=1}^{n}(x_{t_i}-x_{t_{i-1}})'a^{-1}(x_{t_{i-1}})(x_{t_i}-x_{t_{i-1}})$$
$$= \frac{\Delta t}{2}\sum_{i=1}^{n}[g(x_{t_{i-1}})]_{jk}\frac{(x_{t_i}-x_{t_{i-1}})^j}{\Delta t}\frac{(x_{t_i}-x_{t_{i-1}})^k}{\Delta t}$$

$$\underset{(\Delta t\to 0)}{\longrightarrow} \frac{1}{2T}\int_0^1 g_{ij}(x(s))\dot{x}^i(s)\dot{x}^j(s)ds, \quad \left[\dot{x} = \frac{dx}{ds}\right]$$

[using $\Delta t = (\Delta s)T$, so $n\Delta t = n\Delta s T = T \to n\Delta s = 1$]

The Main Theorem (cont)

Lemma: $q(T,S_0,V_0;S_T) \underset{T\to 0}{\approx}$

$$\exp\left\{-\frac{1}{2T}\min_{\substack{x(0)=(S_0,V_0)\\ x(1)=(S_T,free)}}\int_0^1 g_{ij}(x(s))\dot{x}^i(s)\dot{x}^j(s)ds\right\}$$

⇑ Example of a large deviation principle.
Example of a geodesic distance function:

Indeed, Varadhan proved, for D-dimensional diffusions, with $A =$ some set not containing x, that:

$$\boxed{\mathrm{P}_x[X_T \in A] \underset{T\to 0}{\approx} \exp\left\{-\frac{d^2(x,A)}{2T}\right\}},$$

where

$$\boxed{d^2(x,A) = \min_{\substack{\gamma(0)=x\\ \gamma(1)\in A}}\int_0^1 g_{ij}(\gamma(s))\dot{\gamma}^i(s)\dot{\gamma}^j(s)ds}$$

The minimizing paths are geodesics [in a Riemannian space (M,g)]

Notation: Expression(T) $\underset{T\to 0}{\approx} \exp\left\{-\frac{I(parms)}{T}\right\}$
means $I(parms) = -\lim_{T\to 0} T \log \mathrm{Expression}(T)$.
This accounts for many missing factors!

The Main Theorem (cont)

Recall from an earlier slide:

$$C(T, S_0, V_0; K) = e^{-rT} \int_0^{S_T} \max[0, S_T - K]\, q(T, S_0, V_0; S_T) dS_T$$

Since $\dfrac{d^2}{dK^2} \max[0, S_T - K] = \delta(S_T - K)$**, (Dirac delta), we have**

$$\frac{d^2}{dK^2} C(T, S_0, V_0; K) = e^{-rT} q(T, S_0, V_0; K)$$
$$\underset{T \to 0}{\approx} \exp\left\{ -\frac{d^2(x_0, y_0; A_k)}{2T} \right\},$$

using $x_0 = \log S_0, \quad y_0 = V_0,$ **and** $d^2(x_0, y_0; A_k)$
is the geodesic distance to the set A_k **:= the line** $x = k \triangleq \log K$
in the state space (x, y)**.**

With multi-factor (say m **factors) stochastic volatility models, the state is** $(x_t, \vec{\theta}_t) = (x_t, \theta_t^1, \theta_t^2, \cdots, \theta_t^m)$**,**
then $y_0 = \vec{\theta}_0$ **, and** A_k **:= same** $x = k$ **(hyperplane now).**

The Main Theorem (cont)

$$(1)\qquad \frac{d^2}{dK^2}C(T,S_0,V_0;K) \underset{T\to 0}{\approx} \exp\left\{-\frac{d^2(x_0,y_0;A_k)}{2T}\right\}$$

For the Black-Scholes model, with an out-of-the money call (S<K),

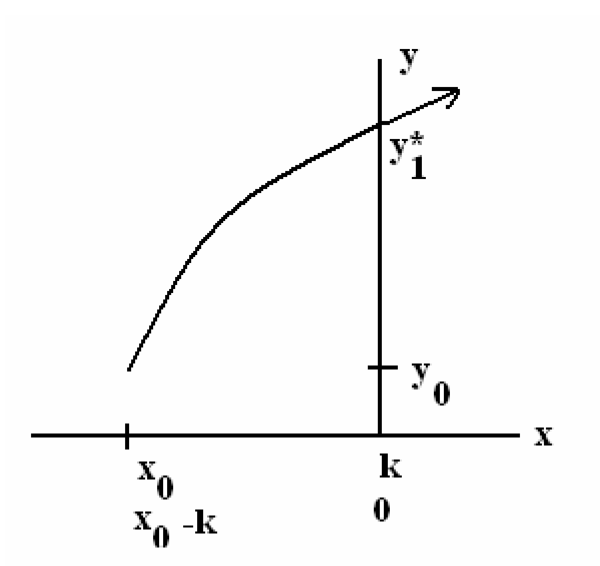
$$\frac{d^2}{dK^2}C_{BS}(T,S_0,V_0;K) \underset{T\to 0}{\approx} \exp\left\{-\frac{(x_0-k)^2}{2V_0T}\right\}$$

Hence, for a general stochastic volatility model,

$$(2)\qquad \frac{d^2}{dK^2}C(T,S_0,V_0;K) \underset{T\to 0}{\approx} \exp\left\{-\frac{(x_0-k)^2}{2\sigma_{imp}^2(x_0-k,y_0)T}\right\}$$

By translation invariance in the x coordinate:

$$d^2(x_0,y_0;A_k) = d^2(x_0-k,y_0;A_0),$$



The Main Theorem (cont)

$$\frac{d^2}{dK^2}C(T,S_0,V_0;K) \underset{T\to 0}{\approx} \exp\left\{-\frac{d^2(x_0-k,y_0;A_0)}{2T}\right\}$$

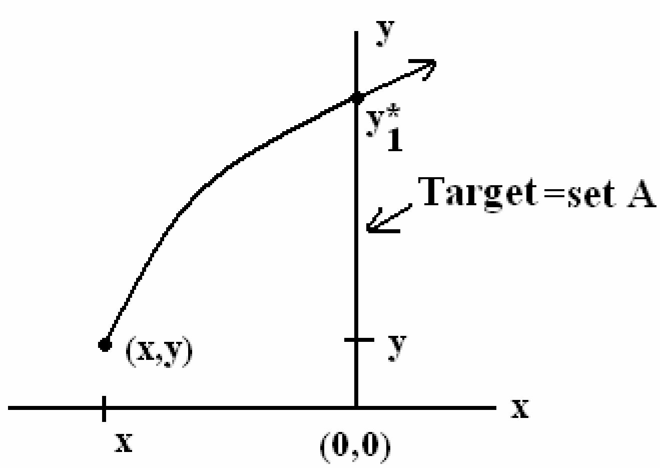
$$\underset{T\to 0}{\approx} \exp\left\{-\frac{(x_0-k)^2}{2\sigma^2_{imp}(x_0-k,y_0)T}\right\},$$

So, comparing, finally yields the main theorem:
<u>Solution to the asymptotic smile problem for diffusions</u>:

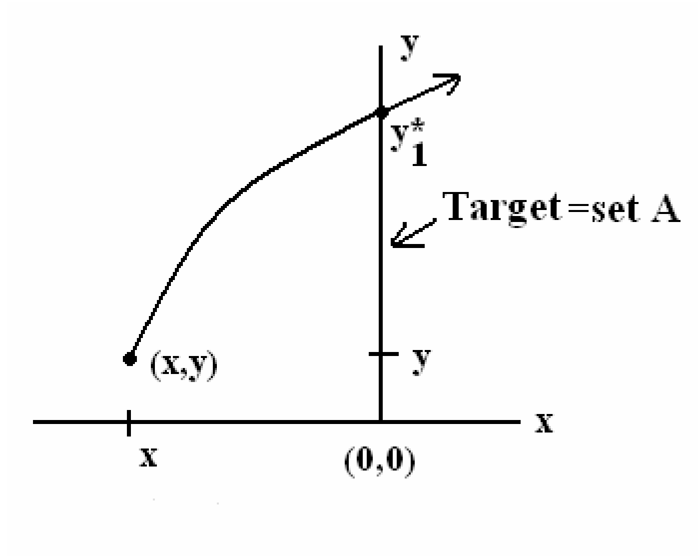
$$\boxed{\sigma^2_{imp}(x,y)=\frac{x^2}{d^2(x,y)}}$$

where $d(x,y)$ = minimum geodesic distance from $P=(x,y)$ to the <u>y-axis</u>. Now x and y are scalar coordinates (recall: the financial variables are $x=\log(S_0/K)$, and $y=V_0$). We have suppressed the dependence on the target set A. The target set is always the y-axis in the remainder of the presentation.

Pictorial solution:(free endpoint/geodesic) problem:



Hitting the Target: the Local Volatility Connection



Given the metric $g = [g_{ij}(x, y)]$, and the starting point $P_0 = (x, y)$, one can compute all the geodesics that pass through P_0. For reasonably close values of x, one of these geodesics will be the distance minimizer to the target. It hits the target at some optimal $y = y_1^*$. It can be shown, although we don't have time today, that

$$
\begin{aligned}
y_1^* &= \lim_{T\to 0} E_{(S_0,V_0)}[V_T \mid S_T = K] \\
&\triangleq \lim_{T\to 0} \alpha(T, S_0, K, V_0) \text{ (the effective local volatility)} \\
&= \lim_{T\to 0} \frac{\int V_T\, p(T, S_0, V_0; K, V_T)\, dV_T}{\int p(T, S_0, V_0; K, V_T)\, dV_T}
\end{aligned}
$$

Effective local volatility (2D problem is equiv. to 1D): $C(T, S, K, V)$ solves exactly, for all T,

$$
C_T = \frac{1}{2}\alpha(T, S, K, V) K^2 C_{KK} - rKC_K
$$

General approaches to computation

We need the distance function $d(x,y)$.
Take the CEV(p)-vol model, for example.
The variance-covariance matrix and the metric are

$$a(x,y) = \left(g^{ij}\right) = \begin{pmatrix} y & \rho y^{p+1/2} \\ \rho y^{p+1/2} & y^{2p} \end{pmatrix}$$

$$g(x,y) = a^{-1} = \left(g_{ij}\right) = \frac{1}{1-\rho^2}\begin{pmatrix} y^{-1} & -\rho y^{-p-1/2} \\ -\rho y^{-p-1/2} & y^{-2p} \end{pmatrix}$$

Here are three general methods.

Method (1): Compute all the geodesics $\left\{\gamma^i(\tau)\right\} = \{X(\tau), Y(\tau)\}$ leaving (x,y). The geodesic equations are well-known:

$$\frac{d^2\gamma^i}{d\tau^2} + \Gamma^i_{jk}(\gamma)\frac{d\gamma^j}{d\tau}\frac{d\gamma^k}{d\tau} = 0, \quad (i = 1,2)$$

where the Christoffel symbol components are given by

$$\Gamma^i_{jk}(x) = \frac{1}{2}g^{im}(x)\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right)$$

General approaches to computation (Geodesic method)

There are two constants of the motion:

(i) <u>kinematic condition</u>: $\frac{(\dot{X})^2}{Y} + \frac{(\dot{Y})^2}{Y^{2p}} = 1$

(ii) <u>conserved x-momentum</u>: $\dot{X} = \pm\sqrt{k}\,Y \;\; (k = \text{const})$

Note on the latter: Whenever the metric is independent of a coordinate x^i, there is a conserved momentum: $U_i = g_{ij}U^j = g_{ij}\dot{\gamma}^j$. In our case, the metric has no x-dependence and so $U_1 = g_{11}\dot{\gamma}^1 = \dot{X}(\tau)/Y(\tau) = \text{const}$, (taking transformed orthogonal coordinates or $\rho = 0$).

Thus, there is a one parameter family of geodesics from (x, y) to the target. One of these ($k = k^*$) is the distance minimizer. The main complication is that the vertical distance to "infinity" is bounded for $p > 1$. Moreover, this vertical move can sometimes be the shortest way to the target. It is straightforward to show:

<u>Theorem (Point-to-target distance bound):</u>
Consider the standardized base point $P_0 = (x,1)$.
Then, under the CEV(p)-vol metric,

$$\boxed{d(x,1) \le \tfrac{1}{p-1} < \infty, \;\; (p > 1 \text{ and } |\rho| < 1)}$$

This bound reflects moving along a vertical geodesic to ∞

General approaches to computation (Geodesic method)

CEV(p)-vol model solution ($\rho = 0,\ p \in R$)

First, define the function $F_p(k) \triangleq B_{1-k}(\frac{1}{2}, 1-p)$,

where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}\,dt$ **(Incomplete Beta).**

I use $G_p(k) = k^{p-3/2} F_{p-1}(k)$, **and** $H_p(k) = k^{p-1} F_p(k)$.

Basic Solution System ($-\infty < p < \frac{3}{2}$)

Step I: set $z = |x| y^{p-3/2}$ **and**

solve for the root $k = k(z)$ **that solves:**

$z = G_p(k)$.

Step II: Then $d(x,y) = y^{1-p} H_p(k)$

Modified Solution System ($\frac{3}{2} \le p < \infty$)

First, calculate the critical values pair:

$$
\begin{aligned}
k_{crit} &= \max_{0 \le k \le 1} \left\{ k : H_p(k) = 1/(p-1) \right\} \\
z_{crit} &= G_p(k_{crit})
\end{aligned}
$$

Then, if $z < z_{crit}$, **use the Basic Solution System, otherwise:**

$$
d(x,y) = y^{1-p} \times \begin{cases} H_p(k), & (z < z_{crit}) \\ 1/(p-1), & (z \ge z_{crit}) \end{cases}
$$

General approaches to computation (Eikonal eqn)

Method (2): Solve a generalized Eikonal problem.
Abbreviating $\partial_i d \equiv \partial d / \partial x^i$, where $x^1 = x$ and $x^2 = y$, the Eikonal/Hamilton-Jacobi eqn is:
$a^{ij}(\partial_i d)(\partial_j d) = 1$ with bound cond: $d(x=0, y) = 0$

i.e. $$\boxed{y d_x^2 + 2\rho y^{p+1/2} d_x d_y + y^{2p} d_y^2 = 1}$$

This is the <u>fastest way to a number</u>.
The trick is to note that there is a scaling form solution:

$$\boxed{d(x,y) = y^{1-p} F(z), \;\; \text{where} \;\; z = x\, y^{p-3/2}}$$

This yields, using $\alpha = p - 3/2$, the non-linear ODE:

$$\boxed{\left[1 + 2\rho\alpha z + \alpha^2 z^2\right](F')^2 + 2(1-p)[\rho + \alpha z]FF' + (1-p)^2 F^2 = 1}$$

Easily solved numerically in Mathematica --
use the StoppingTest option to handle the critical z-values, where $F = 1/(p-1)$. This ODE also forms the <u>starting point</u> for a quasi-analytic solution for general (ρ, p) that extends the $\rho = 0$ solution given in these slides. For details, see:

"Option Valuation under Stochastic Volatility: Volume II". Finance Press, Newport Beach (2007 forthcoming).

General approaches to the computation (Char. Func.)

Method (3): Take a limit with a characteristic function.
If you already know the characteristic function

$$\Phi(T, z, V_0) = E_{(S_0, V_0)}[e^{iz \log(S_T / S_0)}],$$

you can rescale it and find $d(x, y)$ from a Legendre transform/saddle point.

This does not help us directly with the *general* CEV(p)-vol model, as the characteristic functions are known only for half-integers:

$$p = \frac{1}{2}, 1, \frac{3}{2} \text{ (Heston,GARCH/SABR,} \frac{3}{2} \text{model)}$$

<u>But</u>, multi-factor Heston-type models are common in finance, and this may be the most the most direct route to the asymptotics for those applications.
(Details: see Martin Forde's recent paper: arXiv:math.PR/0609117)