USC Math. Finance
April 22, 2003

"Path-dependent Option Valuation under Jump-diffusion Processes"

Alan L. Lewis

These overheads to be posted at www.optioncity.net (Publications)

Topics

Why jump-diffusion models?

Review of the vanilla Euro-style Valuation

A "Standard Machine" for Path-dependent Options:

- a. **Perpetual American**
- **b. Knock-outs (Down-and-out call, etc.)**
- **c. One-touch Options (Rebate term)**

Why Jump-Diffusion Models for Options?

I. Benchmark model (exponential Brownian motion):

Attractive features:

- **limited liability stock prices**
- **uncorrelated, level independent returns**
- **simple formulas (methods) for option prices (euro,amer)**

Weak points:

- **Actual stock price distributions have wider tails**
- **Lacks volatility clustering (auto-corr. of absolute returns)**
- **Lacks stock price jumps**
- **Poor fit to real-world option prices (smile/skew)**

II. Jump-diffusion processes (exponential Lévy processes)

- Stationary, independent increment processes
- Continuous-time analog of Random Walk
- Brownian motion plus Poisson-driven jump process

Attractive features:

- **all the attractive benchmark features +**
- **large flexible class of models, each with a few parameters**
- **wide return tails common (exponential decay, moments)**
- **Good fits to expiring options (fear of jumps/crashes?)**

Weak points:

- **Lacks volatility clustering (auto-corr. of absolute returns)**
- **Brownian motion $\approx$ Large number of small jumps**

Stock price Evolution and Examples

$$S_t = S_0 \exp(X_t),$$

(Assumption: this is under the martingale pricing measure Q)

where $X_t = ct + \sigma B_t + \Delta X_t; \qquad \Delta X_t = \sum_{i=1}^{n(t)} y_i$

Jump probability(t) $\approx \lambda \Delta t$ $\quad y \sim p(y)$ **(Jump distribution)**

Examples of Jump distributions:

(A.1) Merton's 1976 jump-diffusion model with log-normally distributed jumps:

$$p(y) = \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-(y-\mu_J)^2 / 2\delta^2\right]$$

(A.2) Degenerate Case: Point-jump:

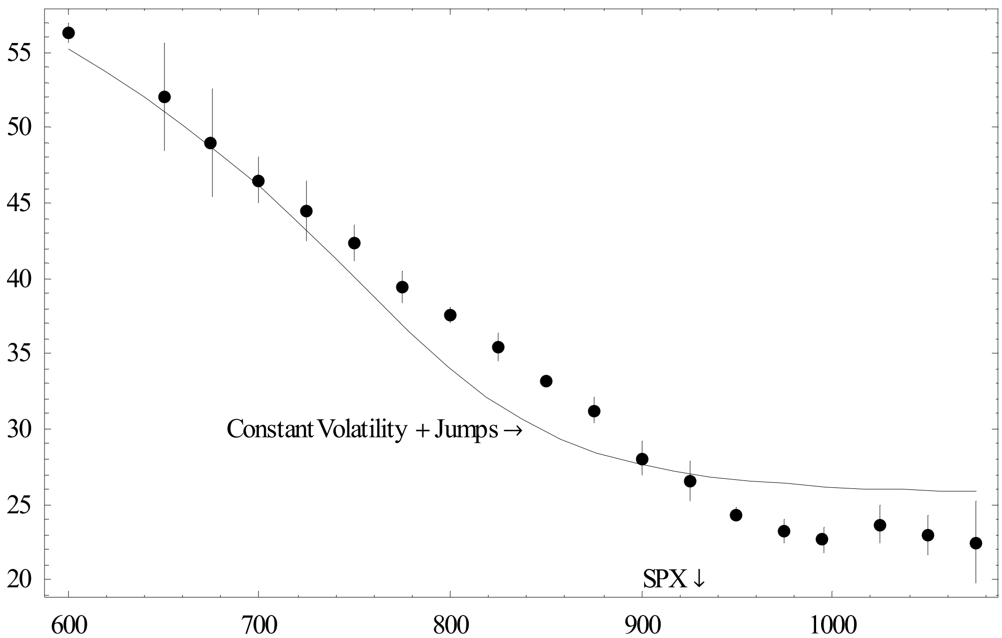
$$p(y) = \delta(y - \mu_J)$$

Typical SPX Smile Fit:

$$\lambda \approx 0.3, \quad \mu_J \approx -0.25, \quad \delta \approx 0.10$$

Figure 1

SPX Options: Implied Volatility vs. Strike on Aug. 16, 2002 (1 month to Expiration)



Vanilla European-style options
Solutions in "Fourier" space are simple

Ingredients:

1. The generalized Fourier transform of the payoff function:
For the call option:

$$\hat{g}(z) = \int_{-\infty}^{\infty} e^{-izx}(e^x - K)^+ dx = -\frac{K^{1-iz}}{(z^2 + iz)}, \quad \operatorname{Im} z < 1$$

2. The characteristic function of the Lévy process:

$$\varphi_T(z) = \int_{-\infty}^{\infty} e^{izx} p_T(x) dx = \mathbb{E}[\exp(izX_T)] = \exp(-T\Psi(z))$$

where $\Psi(z)$ is the "characteristic exponent".
For the Point Jump model:

$$\psi(z) = -iz\omega + \frac{1}{2}z^2\sigma^2 - \lambda\{\exp(i\mu_J z) - 1\} \quad \text{(Entire)}$$

3. Finally, the Call Option price is given by :

$$C(S_0, K, T) = -\frac{e^{-rT}}{2\pi} K \int_{\operatorname{Im} z < -1} \left(\frac{S_0}{K}\right)^{iz} \frac{e^{-T\psi(z)}}{z(z+i)} dz,$$

The integration is along a line parallel to the real z-axis.

European-style options (cont.)

The solution is very easy to derive and "obvious":

First, we need the inversion formula for the payoff function:

$$g(x) = \frac{1}{2\pi}\int_{i\nu-\infty}^{i\nu+\infty} e^{izx}\hat{g}(z)dz, \quad x = \log S_T, \quad z \in \text{Payoff strip}$$

Then, by martingale pricing:

$$C(S_0) = e^{-r\tau}\mathbb{E}[\,g(\log S_T)\,] = \frac{e^{-r\tau}}{2\pi}\mathbb{E}\left[\int_{i\nu-\infty}^{i\nu+\infty} (S_T)^{iz}\,\hat{g}(z)dz\right]$$

$$= \frac{e^{-r\tau}}{2\pi}\mathbb{E}\left[\int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz}\,e^{izX_T}\,\hat{g}(z)dz\right]$$

$$= \frac{e^{-rT}}{2\pi}\int_{i\nu-\infty}^{i\nu+\infty} (S_0)^{iz}\,e^{-T\psi(z)}\hat{g}(z)dz \ ,$$

Ok to exchange the integrations (sufficient conditions) if:

1. $g(x)$ **is Fourier integrable in some Payoff strip** S_g **and bounded for** $|x| < \infty$.
2. $\psi(z)$ **is regular in some strip** S_X: $\alpha < \text{Im}\, z < \beta$
3. $\nu = \text{Im}\, z$ **lies in the intersection of these two strips**

A Standard Machine:
the Down-and-out Call (or Down-and-out "anything")

How we will do it.

1. Write the payoff in terms of its Fourier Transform
2. Write the barrier condition using a representation for $1_{\{Y>0\}}$.
3. Bring an expectation inside an integral.
4. Find a "Fluctuation Identity" to do the expectation.
5. Done with General Formula!

This general procedure works for all the problems I listed at the beginning and probably lots of others.

It saves having to learn a lot of the "heavy machinery" of The Boyarchenko/Levendorskii approach.

6. Then, for your particular model:
Try to do as many integrals as possible analytically; (Residue Calculus).

7. Do the remaining integrals numerically.

Down-and-out Call (or Down-and-out "anything") (cont).

1. Write the payoff in terms of its Fourier Transform

Minimum Process: $\underline{S}_T = \min_{0 \le t \le T} S_t$

$$C_{DOC}(S_0, K, H, T) = e^{-rT} E\left[(S_T - K)^+ 1_{\underline{S}_T > H} \right]$$

or, with $x = \log S_0$, $h = \log H$, **and** $N_T = \min_{0 \le t \le T} X_t$

$$f_{DOC}(x, T) = e^{-rT} E\left[(e^{x + X_T} - K)^+ 1_{N_T > h - x} \right]$$

$$= e^{-rT} E\left[\int_{\operatorname{Im} z < -1} \frac{dz}{2\pi} \exp\{ iz(x + X_T) \} \hat{g}(z) 1_{N_T > h - x} \right]$$

Down-and-out Call (or Down-and-out "anything") (cont)

2. Write the barrier condition using a representation for $1_{\{Y>0\}}$.

$$1_{Y>0} = \int_{\operatorname{Im}\xi<0} \frac{d\xi}{2\pi i\xi} \exp(i\xi Y),$$

where $Y = x - h + N_T$.

This produces:

$$f_{DO}(x,T) = e^{-rT} \times$$

$$E\left[\int_{\operatorname{Im}z<-1} \frac{dz}{2\pi} \int_{\operatorname{Im}\xi<0} \frac{d\xi}{2\pi i\xi} \hat{g}(z) \exp\big(izx + i\xi(x-h)\big) \exp\big(izX_T + i\xi N_T\big) \right]$$

3. Bring an expectation inside an integral.

We need a formula for $E\left[\exp\left(izX_T + i\xi N_T\right)\right]$.

4. Find a "Fluctuation Identity" to do the expectation

Some Fluctuation Identities:

1. Factorization identities: (Spitzer, Rogozin, others)

$$\frac{q}{q+\psi(z)} = \phi_q^+(z)\phi_q^-(z),$$

where

$$\phi_q^+(z) = E\left[\exp\left(izM_{\tau(q)}\right)\right] = q\int_0^\infty e^{-qt} E\left[e^{izM_t}\right]dt$$

$$\phi_q^-(z) = E\left[\exp\left(izN_{\tau(q)}\right)\right] = q\int_0^\infty e^{-qt} E\left[e^{izN_t}\right]dt$$

$\tau(q)$ **is an independent (exponentially distributed) random stopping time.**

Not computationally effective. But, this one is:

$$\phi_q^-(\xi) = \exp\left\{ \int_{(\mathrm{Im}\,\xi)^+ < \mathrm{Im}\,\eta < \sigma^+} \frac{d\eta}{(-2\pi i)} \frac{\xi \log[q+\psi(\eta)]}{\eta(\xi-\eta)} \right\}$$

4. Find a "Fluctuation Identity" to do the expectation

Some Fluctuation Identities:

2. Here's the one we really need:

$$q\int_0^{\infty} e^{-qT} E[\exp(izX_T + i\xi N_T)]dT = \phi_q^+(z)\phi_q^-(\xi + z)$$

By Laplace Inversion:

$$E[\exp(izX_T + i\xi N_T)] = \int_{\mathrm{Re}\,q>r} \frac{dq}{2\pi iq} e^{qT}\, \phi_q^+(z)\phi_q^-(\xi + z)$$

5. Done with General Formula!

Here it is:

$$f_{DO}(x,T) = \int_{\mathrm{Re}\,q>r} \frac{dq}{2\pi i} e^{qT} F_{DO}(x,q),$$

where for the call option payoff, it reads

$$F_{DOC}(x,q) = \left(\frac{Ke^{-rT}}{q}\right)\times$$

$$\int_{C_1} \frac{d\xi}{2\pi i} \int_{C_2} \frac{dz}{2\pi} \frac{\exp\{iz(h-k)+i\xi(x-h)\}}{(z-\xi)z(z+i)} \phi_q^+(z)\phi_q^-(\xi)$$

Integration contours:

$C_1 : \mathrm{Im}\,\lambda^- < \mathrm{Im}\,\xi < -1,$

$C_2 : \mathrm{Im}\,\alpha_1(q) < \mathrm{Im}\,z < -1$ **and** $\mathrm{Im}\,\xi < \mathrm{Im}\,z$

6. Then, for your particular model:
Try to do as many integrals as possible analytically;
(Residue Calculus).

7. Do the remaining integrals numerically.

<u>Example 1</u>: Suppose your model has "No negative jumps".
(This means the Barrier is crossed continuously).

When there are no negative jumps, the Laplace transform of the down-and-out call option is given by: ($x > h$)

$$F_{DOC}(x,q)\big|_{NNJ} = F_E(x,q) - \exp\{\, i\gamma(q)(x-h)\,\}\, F_E(h,q),$$

where $F_E(x,q)$ is the Laplace transform of the European (no barrier) call option:

$$F_E(x,q) = \int_{\operatorname{Im}\alpha_1(q) < \operatorname{Im} z < -1} \frac{dz}{2\pi} \frac{-Ke^{-rT}}{z(z+i)(q+\psi(z))} \exp\{\, iz(x-k)\,\}.$$

(This one could be proved directly from the PIDE).

6. **Then, for your particular model:**
   Try to do as many integrals as possible analytically;
   (Residue Calculus).

7. **Do the remaining integrals numerically.**

Example 2: Suppose your model has a "Negative Point Jump".

Then, you have to investigate the roots (zeros) of

$$q + \psi(z) = 0\,,$$

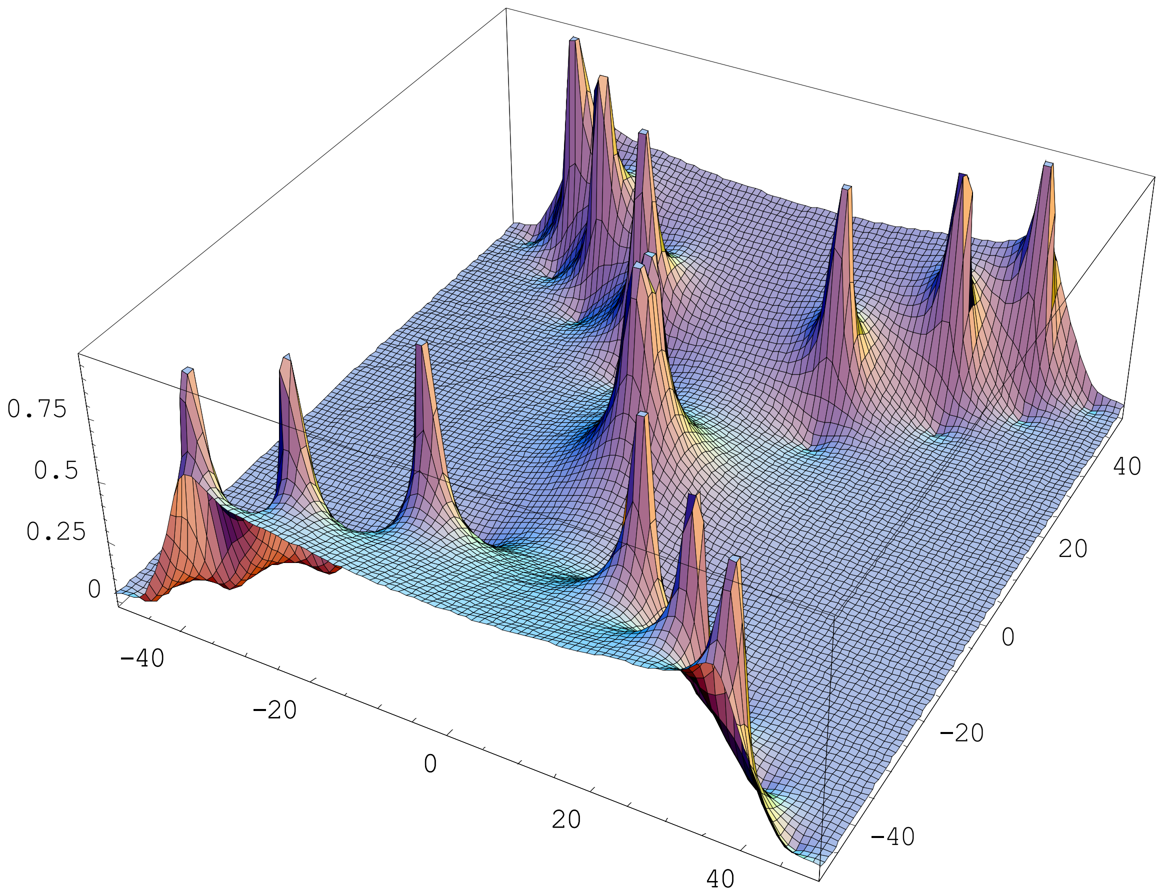
where

$$q + \psi(z) = q - iz\omega + \tfrac{1}{2} z^2 \sigma^2 - \lambda\left(\exp\left(i\mu_J z\right) - 1\right)$$

It turns out there are (almost certainly) an infinity of these in the complex z-plane. Then, I have a (conjectured) result using these roots:

$$F_{DOC}(x,q) = \frac{i\,e^{x-rT}}{\alpha(\alpha+i)}\left(\frac{1}{\psi'(\alpha)} + \lim_{N_\gamma \to \infty} \sum_{i=1}^{N_\gamma} \frac{\exp\{i(\gamma_i - \alpha)(x-h)\}}{\psi'(\gamma_i)}\right)$$

α **is the single lower half-plane root. (Im** $\alpha < 0$**)**
γ_i **are the infinity of upper half-plane roots. (Im** $\gamma_i > 0$**)**



1.

A plot of $|f'(z)/f(z)|$ **where** $f(z) = r + \psi(z)$ **for the Merton jump-diffusion**

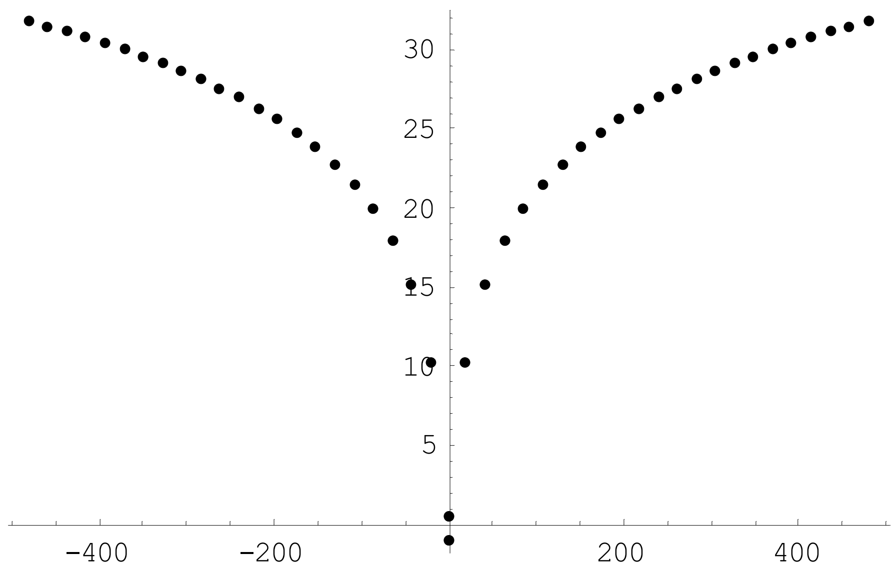


Figure 2.
The location of some roots of $r + \psi(z) = 0$ for the point jump model.